Fractional Signal Processing And Application
(分數信號處理及應用)

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Outline

1. Introduction to Image Processing
2. Introduction to Fractional Signal Processing
3. Image Sharpening Using Fractional Calculus
4. Image Interpolation Using Fractional Delay
5. Image Watermarking Using Fractional Fourier Transform
6. Conclusions

Image Compression

64 K bytes

8 K bytes

Image Interpolation

64 x 64

128 x 128

Image Halftoning

Image Sharpening

Image Inpainting

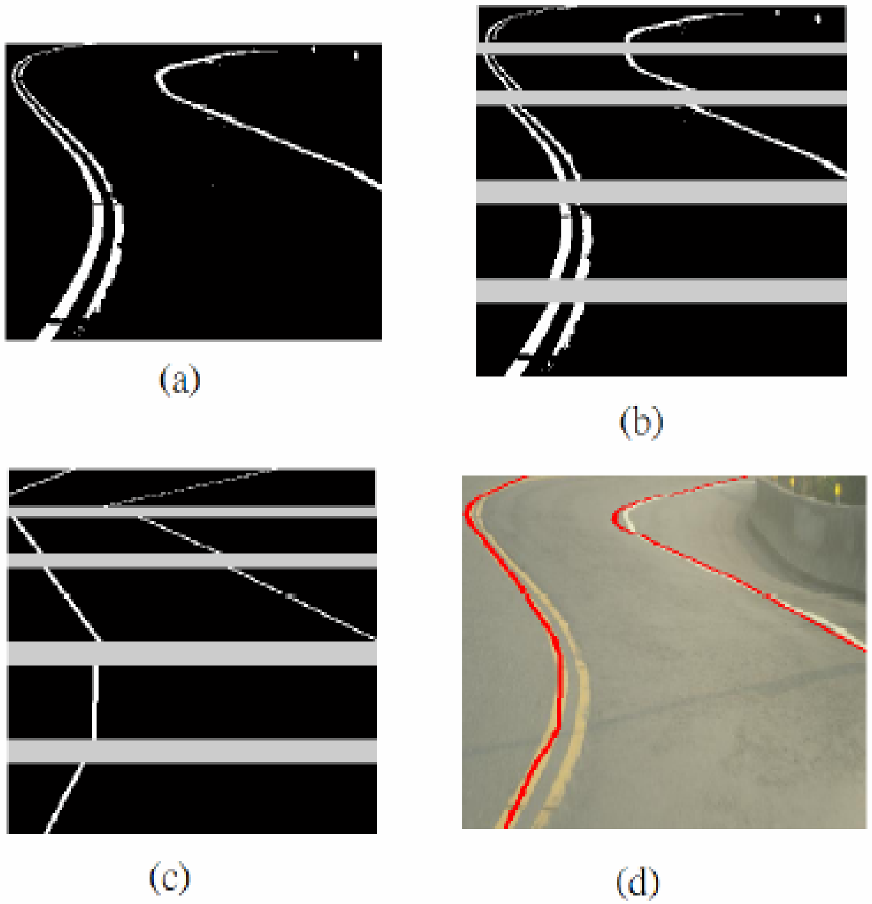
Image Inpainting

Image Matting

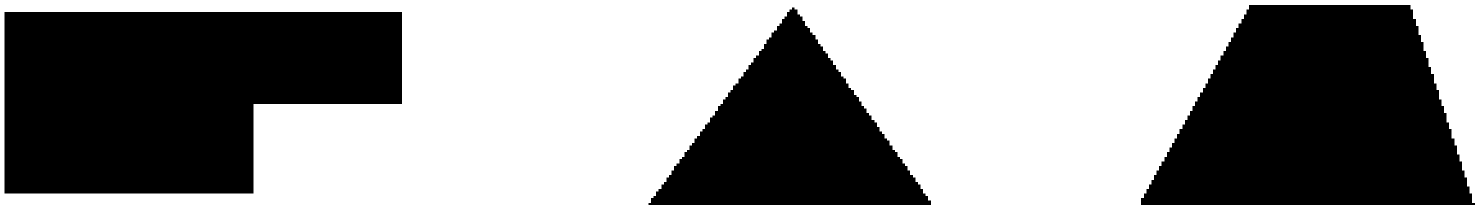
移花　　　　接木

Image Binarization

Lane Detection



Feature Extraction



L-shape:

[0.879 2.862 5.225 4.693 10.339 6.237 9.666]

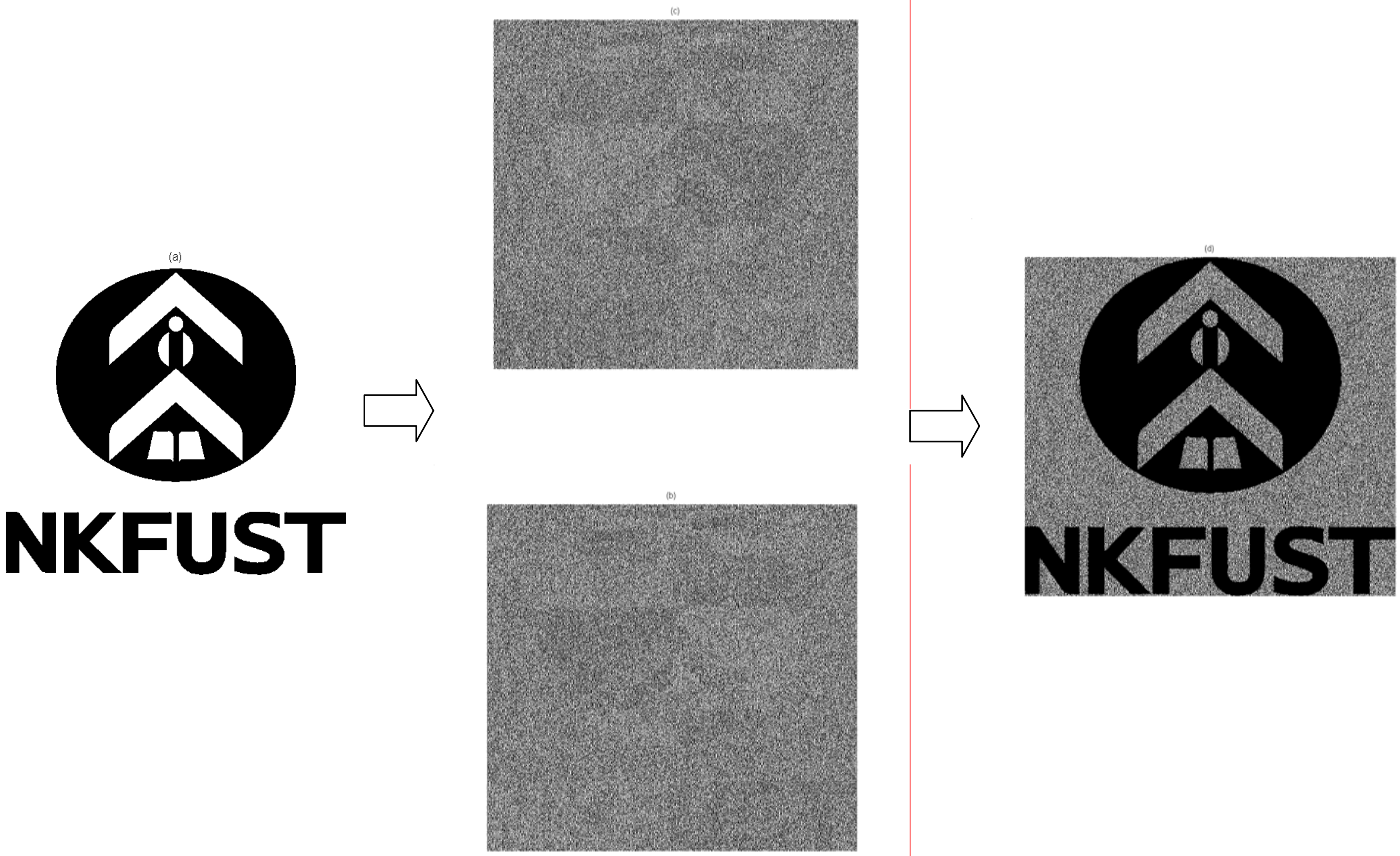
Triangle:

[0.925 2.869 7.937 5.454 12.425 6.894 12.409]

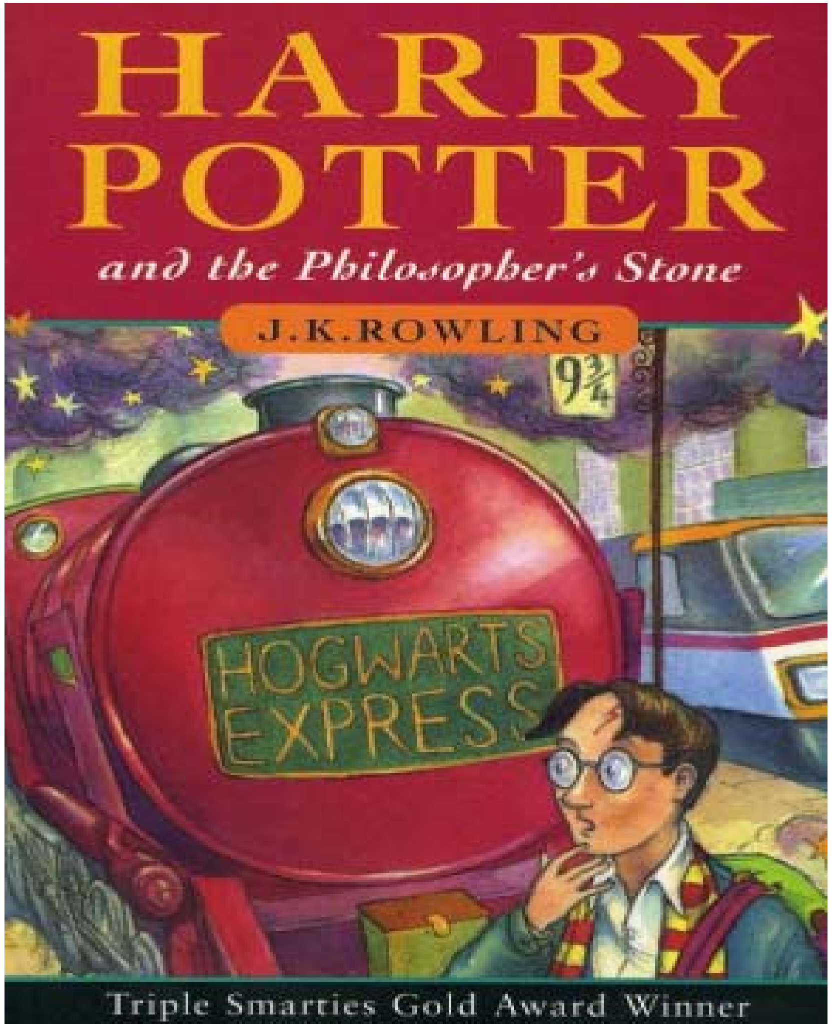
Trapezoid:

[0.918 2.938 6.512 5.874 12.679 7.369 11.928]

Visual Cryptograph



Fractional Platform



— CHAPTER SIX —

The Journey from Platform Nine and Three-Quarters

Harry's last month with the Dursleys wasn't fun. True, Dudley was now so scared of Harry he wouldn't stay in the same room, while Aunt Petunia and Uncle Vernon didn't shut Harry in his cupboard, force him to do anything or shout at him – in fact, they didn't speak to him at all. Half-terrified, half-furious, they acted as though any chair with Harry in it was empty. Although this was an improvement in many ways, it did become a bit depressing after a while.

Harry kept to his room, with his new owl for company. He had decided to call her Hedwig, a name he had found in *A History of Magic*. His school books were very interesting. He lay on his bed reading late into the night, Hedwig swooping in and out of the open window as she pleased. It was lucky that Aunt Petunia didn't come in to hoover any more, because Hedwig kept bringing back dead mice. Every night before he went to sleep, Harry ticked off another day on the piece of paper he had pinned to the wall, counting down to September the first.

Fractional Calculus

$$\frac{d^n y}{dx^n}$$

n: non-integer ?

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} \quad ?$$

paradox

Leibnitz

L'Hopital

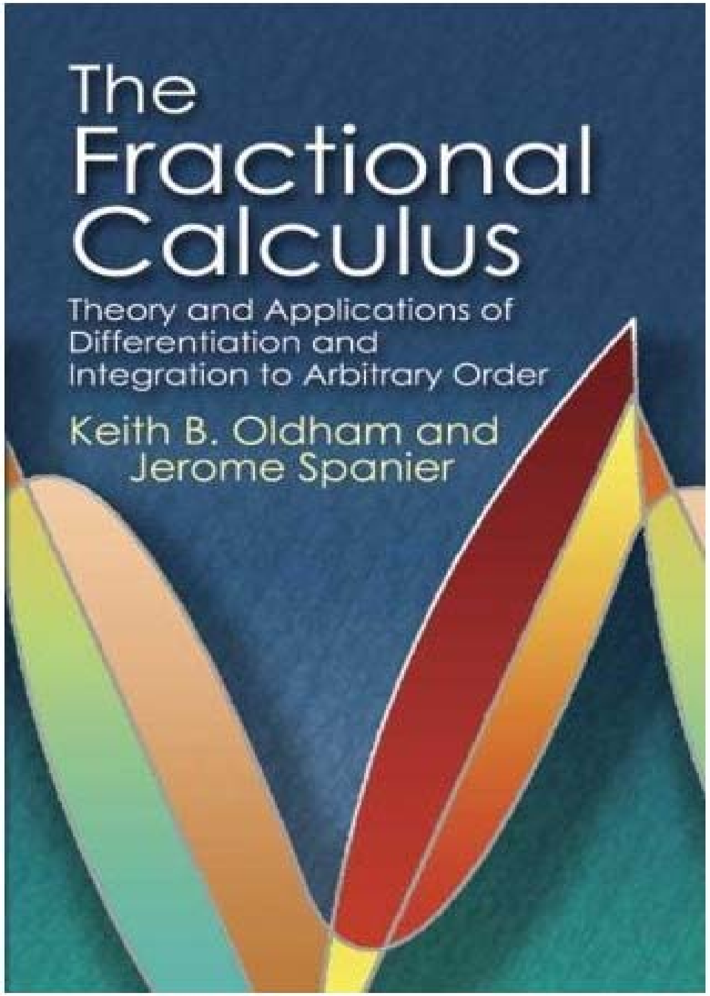
Fractional Calculus

People who have worked on fractional calculus.

- Euler (1730)
- Lagrange (1772)
- Laplace (1812)
- Fourier (1822)
- Riemann (1847)

For almost 300 years the investigation was mostly mathematical. Only during last 30 years the mathematical deductions found some physical and engineering relevance.

Fractional Calculus



1695 G. W. Leibniz, Letter from Hanover, Germany, September 30, 1695 to G. A. L'Hospital. *Leibnizen Mathematische Schriften*, Vol. 2, pp. 301–302. Olms Verlag., Hildesheim, Germany, 1962. First published in 1849.

Leibniz wrote prophetically, "Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt[2]{dx:x}$, an apparent paradox, from which one day useful consequences will be drawn."

1697 G. W. Leibniz, Letter from Hanover, Germany, May 28, 1697 to J. Wallis, *Leibnizen Mathematische Schriften*, Vol. 4, p. 25. Olms Verlag., Hildesheim, Germany, 1962. First published in 1859.

In this letter Leibniz discusses Wallis' infinite product for π. Leibniz mentions differential calculus and uses the notation $d^{\frac{1}{2}}y$ to denote a derivative of order $\frac{1}{2}$.

1730 L. Euler, "De Progressionibus Transcentibus, sev Quarum Termini Algebraice Dari Nequeunt." *Comment. Acad. Sci. Imperialis Petropolitanae* **5**, 38–57 (1738).

1972 T. R. Prabhakar, "Hypergeometric Integral Equations of a General Kind and Fractional Integration." *SIAM J. Math. Anal.* **3**, 422–425.

Some integral equations containing hypergeometric functions in two variables are studied with the use of fractional integration.

1974 To appear in late 1974 or early 1975 B. Ross, "*A Profile of Fractional Calculus*," Chapter V. Ph.D. Thesis, New York Univ., New York.

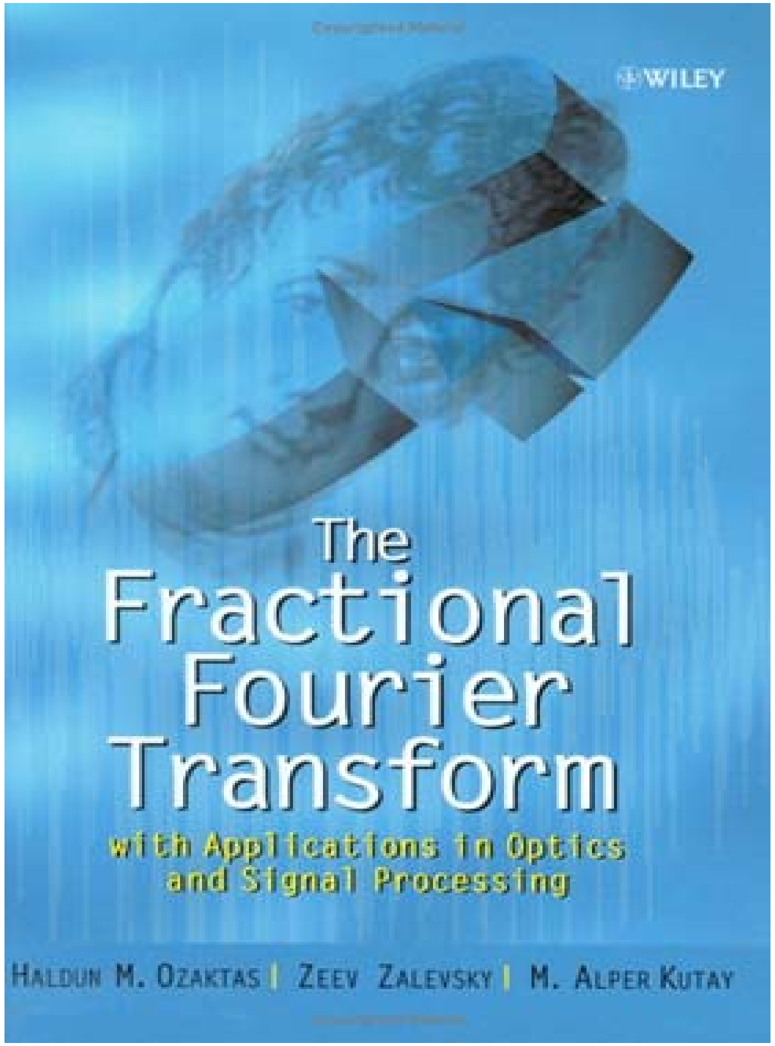
1975 The Proceedings of the University of New Haven Colloquium on Fractional Calculus and Its Applications to the Mathematical Sciences to be held June 1974 is expected to be published late in 1975.

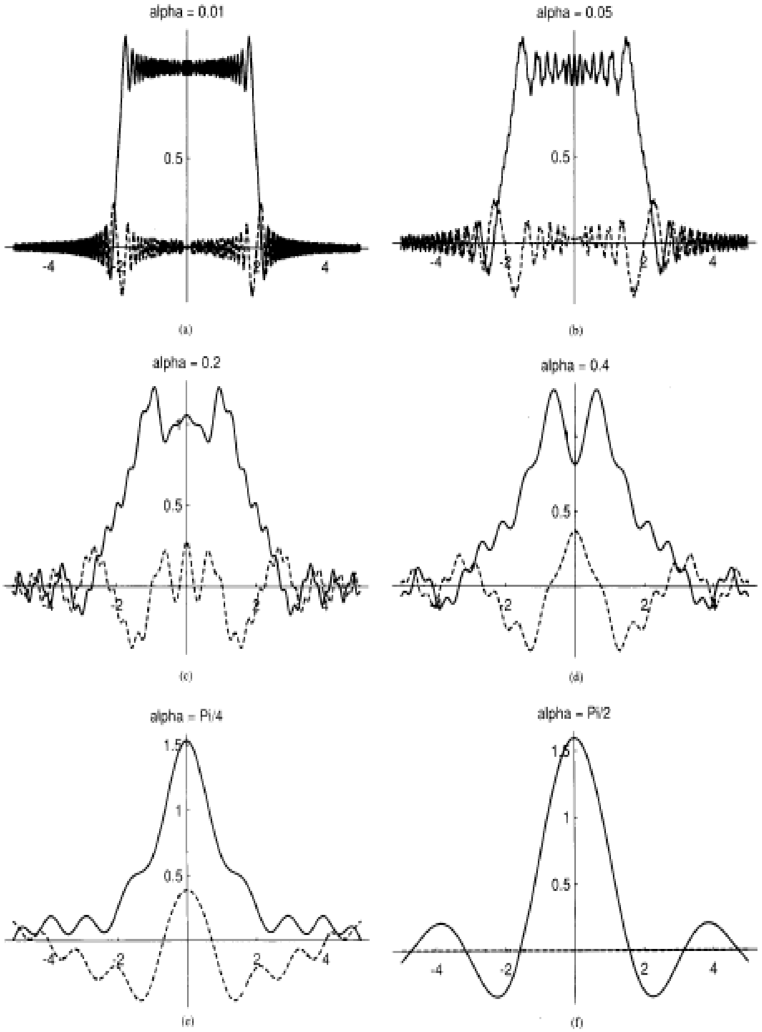
Fractional Calculus

IFAC Workshop on Fractional Differentiation and its Applications

1. First: Bordeaux France, July 19-21, 2004
2. Second: Porto, Portugal, July 19-21, 2006
3. Third: Ankara Turkey, Nov. 2008

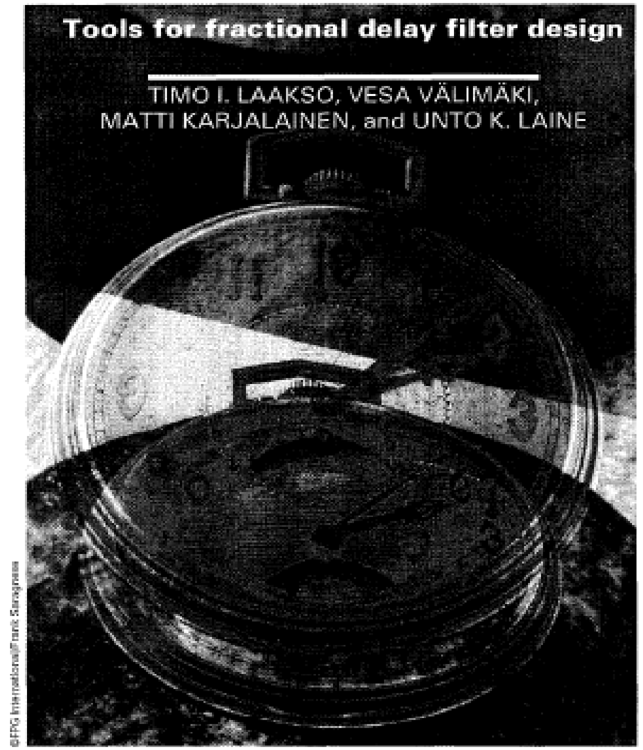
Fractional Transform



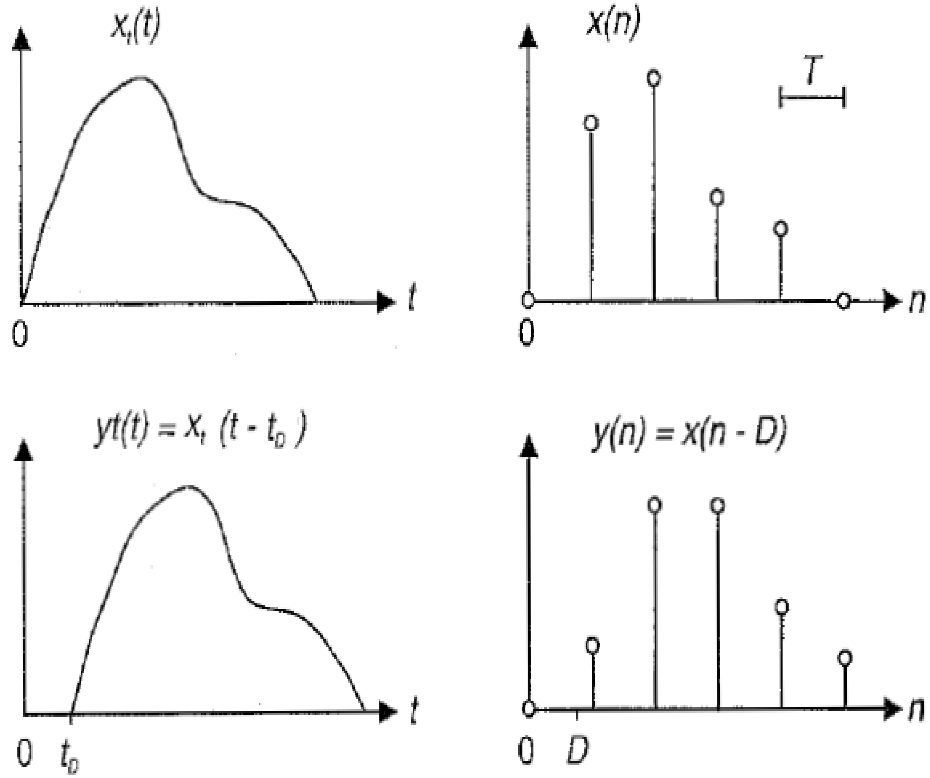


Fractional Delay

Splitting



the Unit Delay



(a) (b)

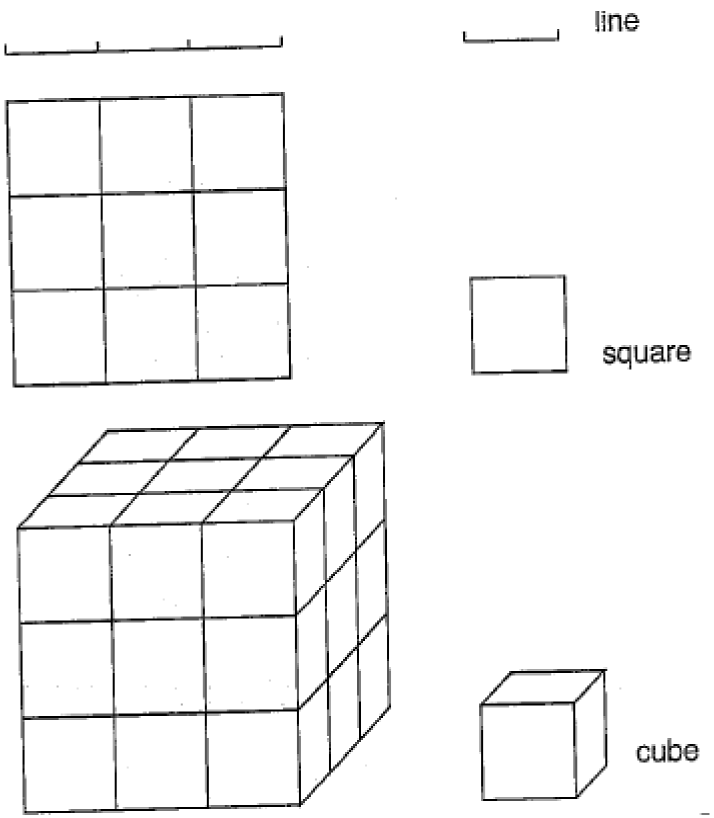
IEEE Signal Processing Magazine (Jan. 1996)

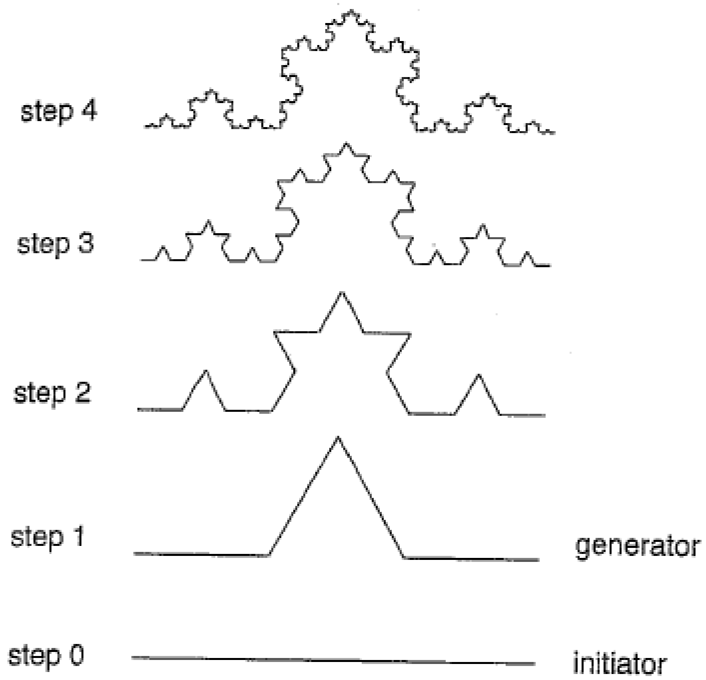
Fractal Dimension

D=1

D=2

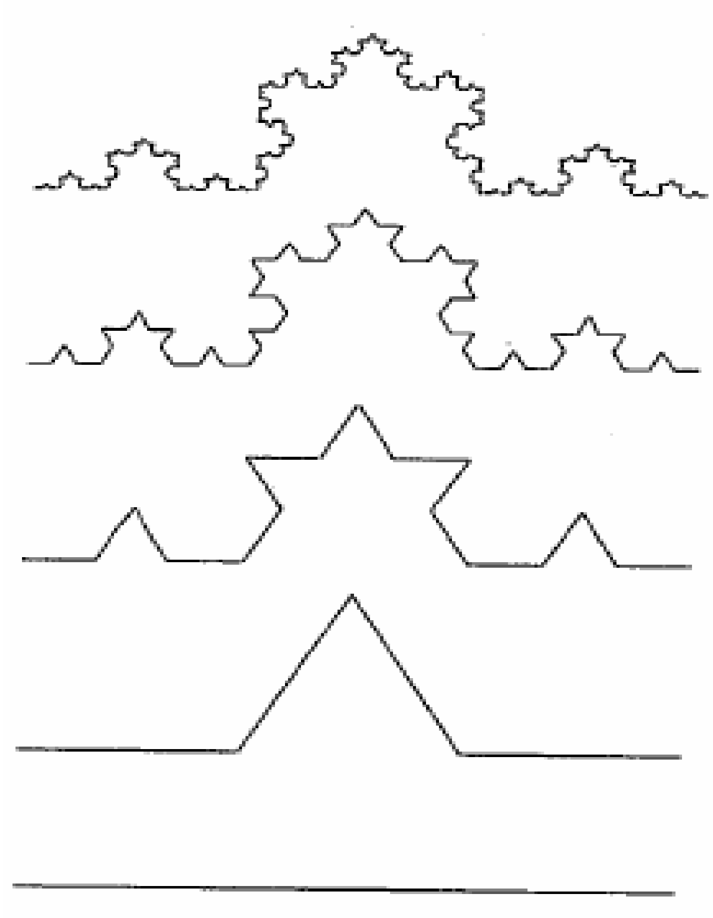
D=3





Koch curve: D=1.2619

Length of Fractal



$$length = (\tfrac{4}{3})^4$$

$$length = (\tfrac{4}{3})^3$$

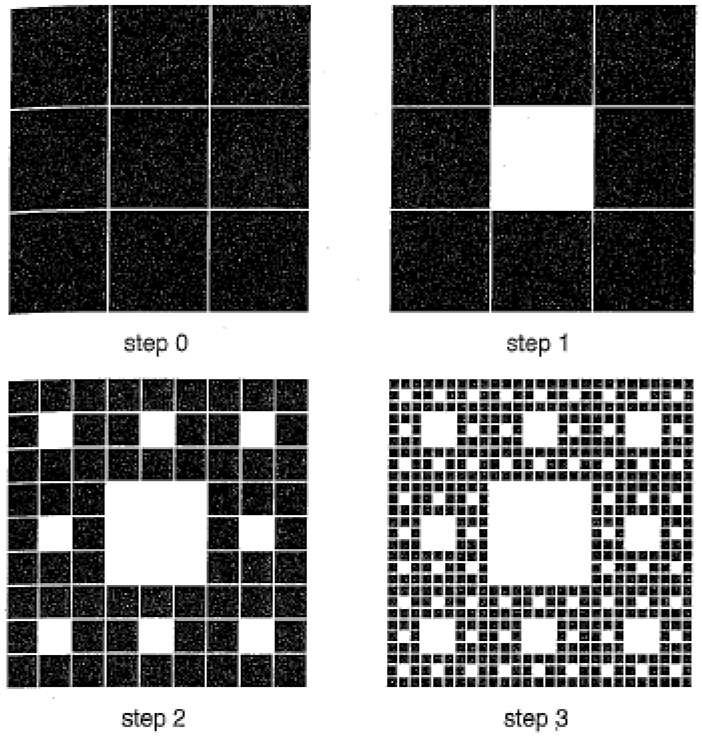
$$length = (\tfrac{4}{3})^2$$

$$length = (\tfrac{4}{3})^1$$

$$length = (\tfrac{4}{3})^0 = 1$$

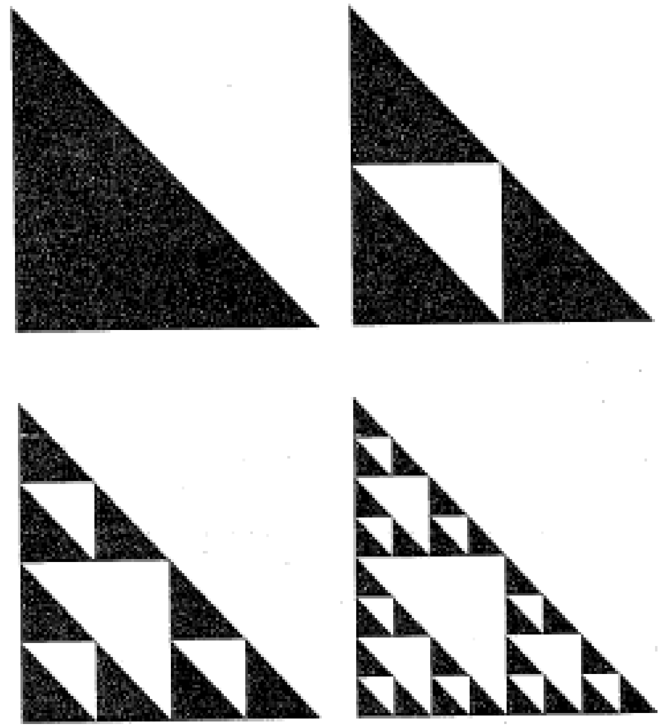
Fractal Dimension

Sierpinski Carpet



D=log(n)/log(s)=log(8)/log(3)=1.8928

Sierpinski Gasket



D=log(n)/log(s)=log(3)/log(2)=1.5850

Fractional Signal Processing

Fractional Operator:

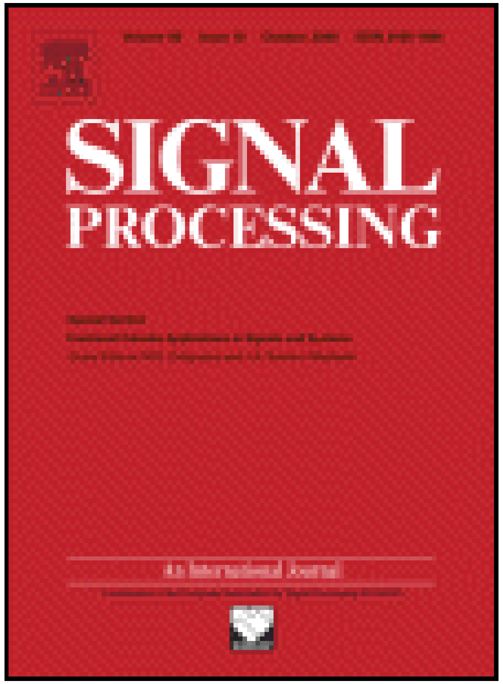
1. Fractional Calculus
2. Fractional Delay
3. Fractional Fourier Transform

Fractional measure:

1. Fractal Dimension
2. Fractional Moment

Fractional Signal Processing

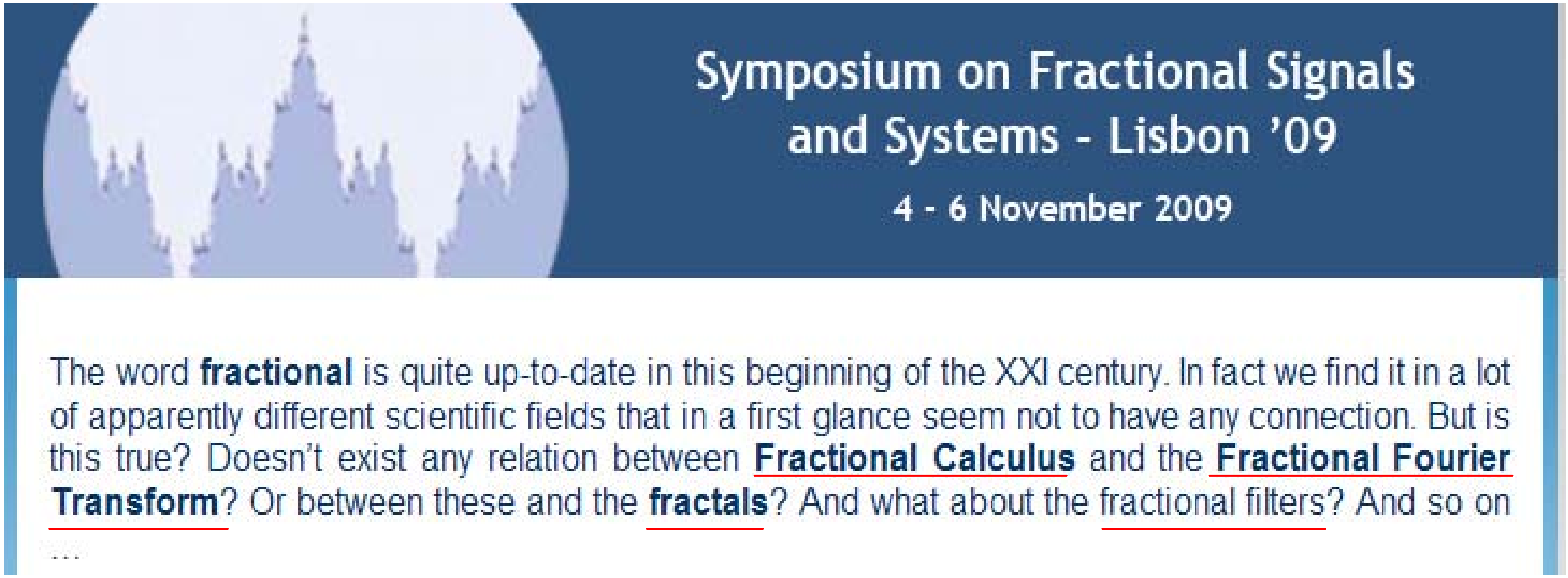
Special Issue:



1. Fractional Signal Processing and Applications, (vol. 83, No.11, Nov. 2003).
2. Fractional Calculus Applications in Signals and Systems (vol. 86, No.10, Oct. 2006).
3. Advances in Fractional Signals and Systems (Apr. 2010)

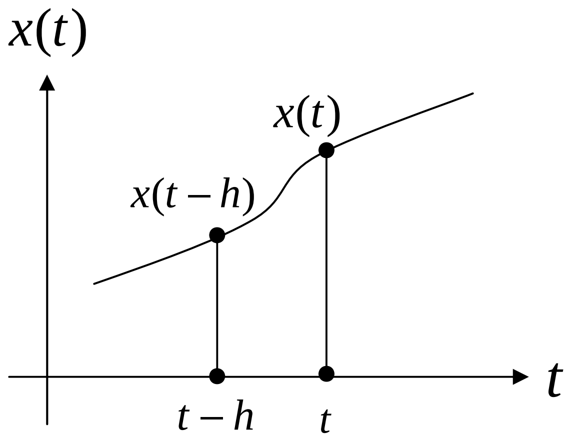
Fractional Signal Processing

Recent symposium: https://www.fct.unl.pt/fss09/



Fractional Calculus - Definition

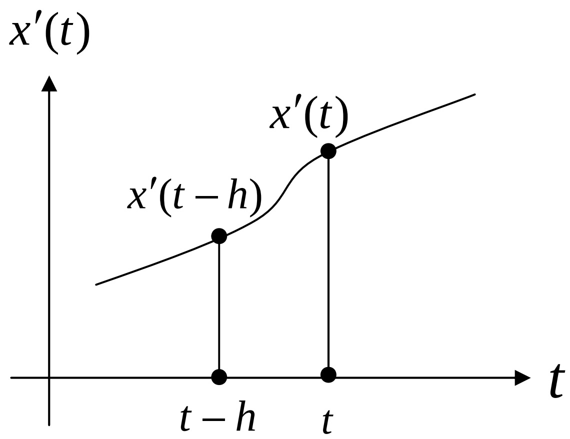
$$x'(t)=\lim_{h\to 0}\frac{x(t)-x(t-h)}{h}$$



$$x''(t)=\lim_{h\to 0}\frac{x'(t)-x'(t-h)}{h}$$

$$=\lim_{h\to 0}\frac{\lim\limits_{h\to 0}\dfrac{x(t)-x(t-h)}{h}-\lim\limits_{h\to 0}\dfrac{x(t-h)-x(t-2h)}{h}}{h}$$

$$=\lim_{h\to 0}\frac{x(t)-2x(t-h)+x(t-2h)}{h^2}$$



Fractional Calculus - Definition

$$x'''(t) = \lim_{h \to 0} \frac{x''(t) - x''(t-h)}{h}$$

$$= \lim_{h \to 0} \frac{\lim_{h \to 0} \frac{x(t) - 2x(t-h) + x(t-2h)}{h^2} - \lim_{h \to 0} \frac{x(t-h) - 2x(t-2h) + x(t-3h)}{h^2}}{h}$$

$$= \lim_{h \to 0} \frac{x(t) - 3x(t-h) + 3x(t-2h) - x(t-3h)}{h^3}$$

Similarly

$$x^{(4)}(t) = \lim_{h \to 0} \frac{x'''(t) - x'''(t)}{h}$$

$$= \lim_{h \to 0} \frac{x(t) - 4x(t-h) + 6x(t-2h) - 4x(t-3h) + x(t-4h)}{h^4}$$

Fractional Calculus - Definition

Define delay operator:

$$x(t) \longrightarrow \boxed{q^{-kh}} \longrightarrow x(t-kh)$$

Then, from the equalities

$$(1-q^{-h})^1 = 1-q^{-h}$$
$$(1-q^{-h})^2 = 1-2q^{-h}+q^{-2h}$$
$$(1-q^{-h})^3 = 1-3q^{-h}+3q^{-2h}-q^{-3h}$$
$$(1-q^{-h})^4 = 1-4q^{-h}+6q^{-2h}-4q^{-3h}+q^{-4h}$$
$$\vdots$$
$$(1-q^{-h})^m = \sum_{k=0}^{m}\binom{m}{k}(-q^{-h})^k = \sum_{k=0}^{m}(-1)^k\binom{m}{k}q^{-kh}$$

Fractional Calculus - Definition

we have

$$x(t) \longrightarrow \boxed{\lim_{h\to 0}\frac{1}{h}(1-q^{-h})^{1}} \longrightarrow \lim_{h\to 0}\frac{x(t)-x(t-h)}{h}=x'(t)$$

$$x(t) \longrightarrow \boxed{\lim_{h\to 0}\frac{1}{h^{2}}(1-q^{-h})^{2}} \longrightarrow \lim_{h\to 0}\frac{x(t)-2x(t-h)+x(t-2h)}{h^{2}}=x''(t)$$

$$x(t) \longrightarrow \boxed{\lim_{h\to 0}\frac{1}{h^{3}}(1-q^{-h})^{3}} \longrightarrow \lim_{h\to 0}\frac{x(t)-3x(t-h)+3x(t-2h)-x(t-3h)}{h^{3}}=x'''(t)$$

$$\vdots$$

$$\Rightarrow \qquad x^{(m)}(t)=\lim_{h\to 0}\frac{(1-q^{-h})^{m}}{h^{m}}x(t)$$

where $m=0,1,2,3,\cdots$

Fractional Calculus - Definition

$$x^{(m)}(t) = \lim_{h \to 0} \frac{(1 - q^{-h})^m}{h^m} x(t)$$

Change integer m
to fractional ν

$$x^{(\nu)}(t) = \lim_{h \to 0} \frac{(1 - q^{-h})^\nu}{h^\nu} x(t)$$

Fractional Calculus - Definition

Fractional derivative:

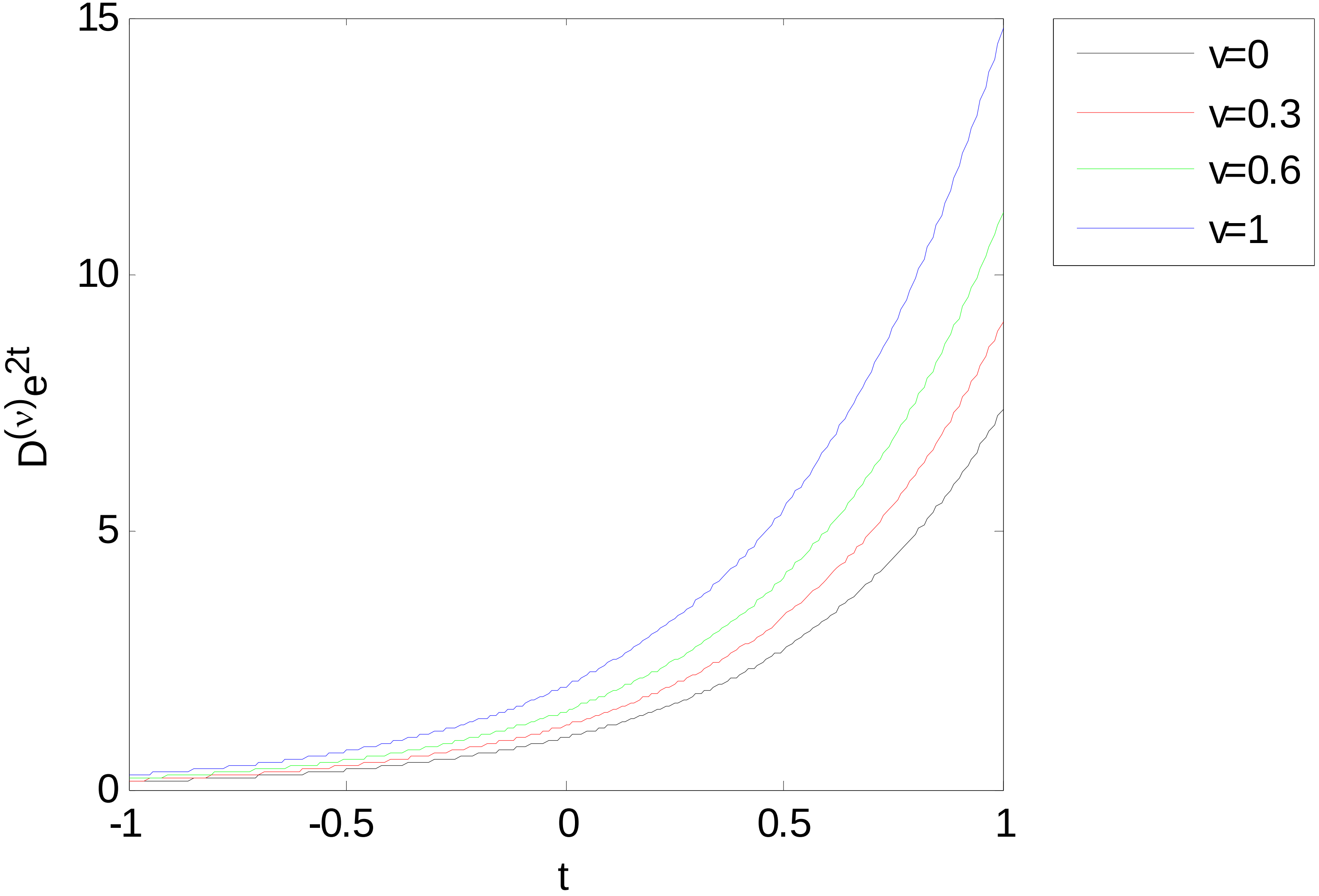
$$x^{(v)}(t) = \lim_{h\to 0} \frac{(1-q^{-h})^v}{h^v} x(t)$$

$$= \lim_{h\to 0} \frac{1}{h^v}\left(\sum_{k=0}^{\infty} (-1)^k \binom{v}{k} q^{-kh}\right) x(t)$$

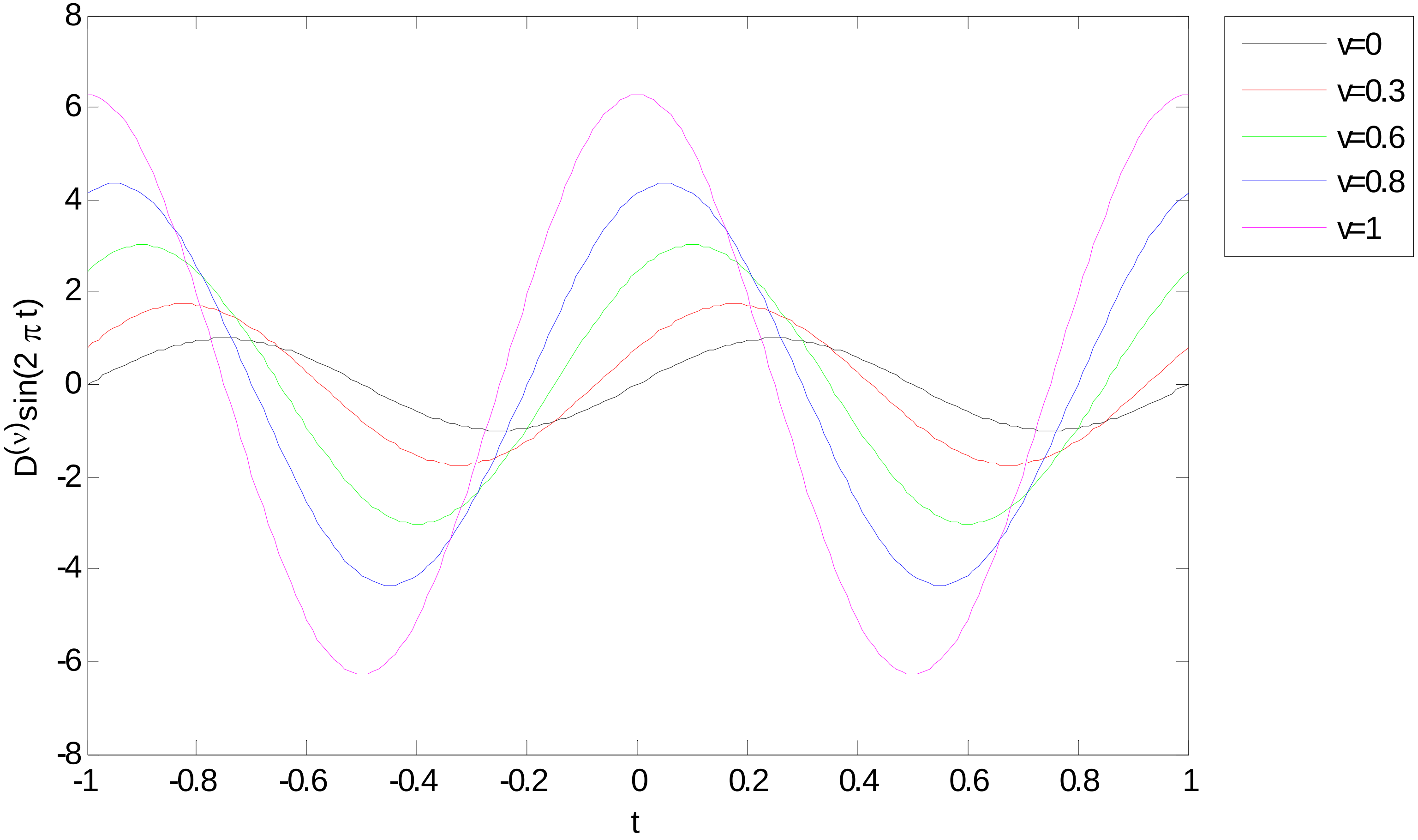
$$= \lim_{h\to 0} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{v}{k}}{h^v} x(t-kh)$$

Grünwald -Letnikov derivative

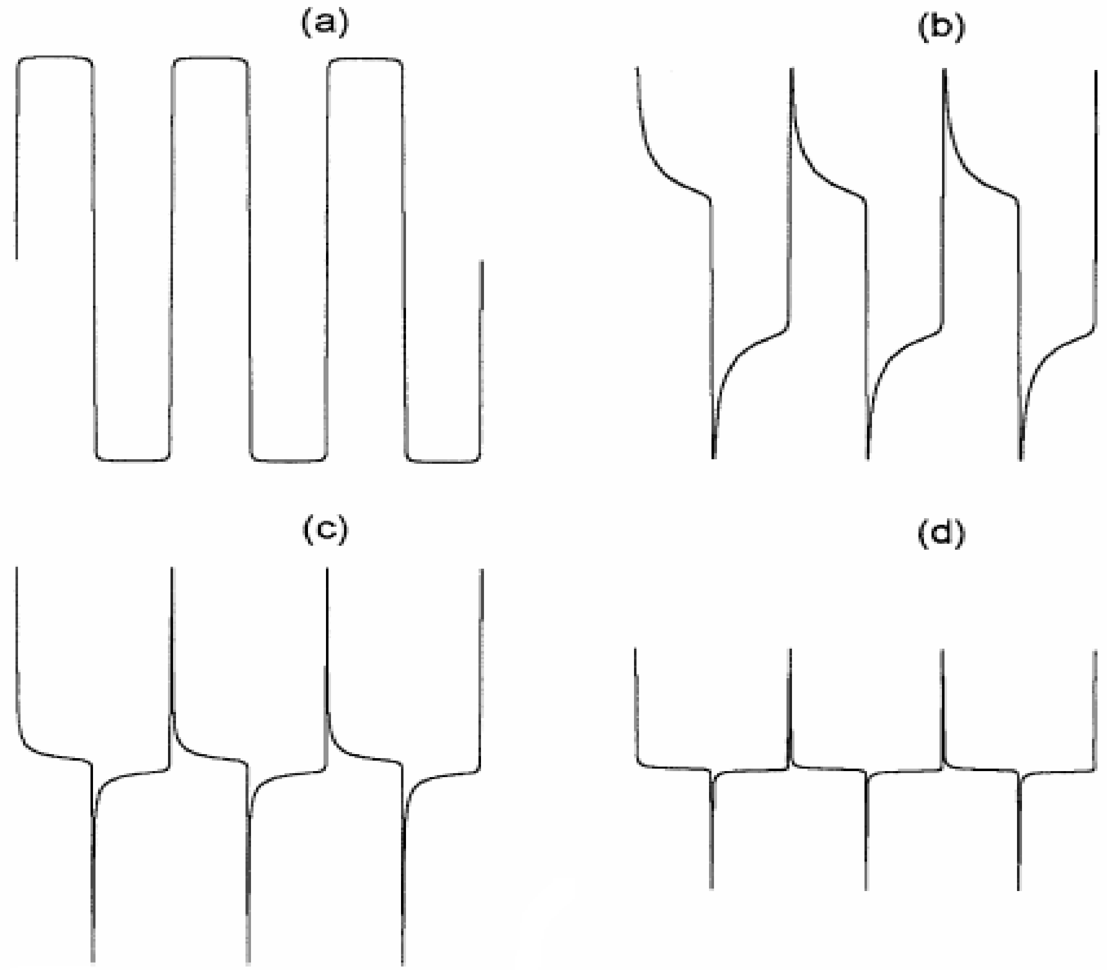
Fractional Calculus - Example



Fractional Calculus - Example



Fractional Calculus - Example



(a) $v = 0$, (b) $v = 0.2$, (c) $v = 0.5$, (d) $v = 0.7$.

Fractional Calculus - Differentiator

Fourier transform pairs:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) \xleftrightarrow[FT]{} X(\omega)$$

Using fractional differential operator D^v ,we have

$$D^v x(t) = D^v \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) D^v [e^{j\omega t}] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) (j\omega)^v e^{j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [(j\omega)^v X(\omega)] e^{j\omega t} dt$$

$$\Rightarrow \qquad D^v x(t) \xleftrightarrow[FT]{} (j\omega)^v X(\omega)$$

Fractional Calculus - Differentiator

Ideal Frequency Response (Continuous-time case)

$$x(t) \longrightarrow \boxed{\begin{array}{c} H_d(\omega) \\ = (j\omega)^v \end{array}} \longrightarrow D^v x(t)$$

Ideal Frequency Response (Discrete-time case)

$$x[n] \longrightarrow \boxed{\begin{array}{l} H_d(\omega) \\ = (j\omega)^v e^{-jp\omega} \end{array}} \longrightarrow D^v x[n-p]$$

p : delay value

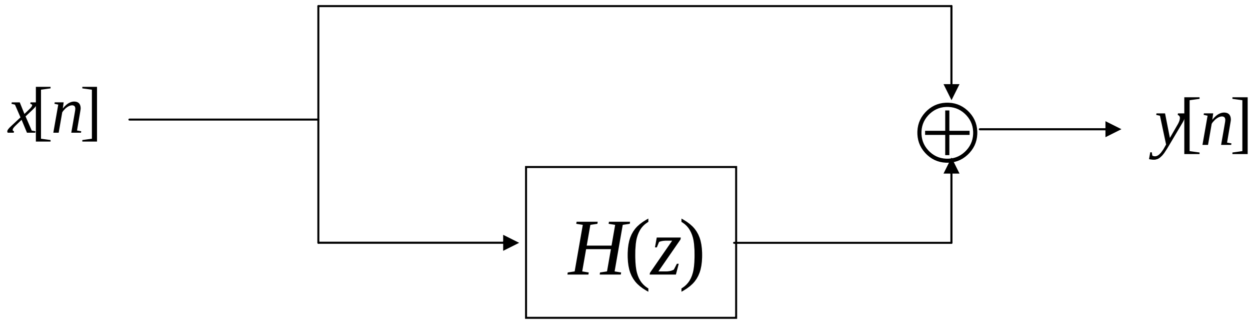
Fractional Calculus-Application

Mach band effect

30	60	90	120	150	180	210

Fractional Calculus - Application

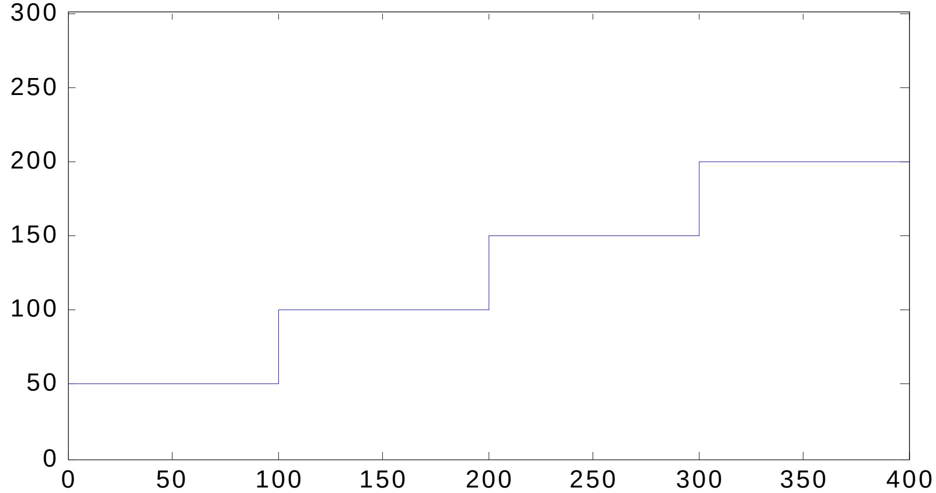
Generate Mach band effect using fractional order differentiator



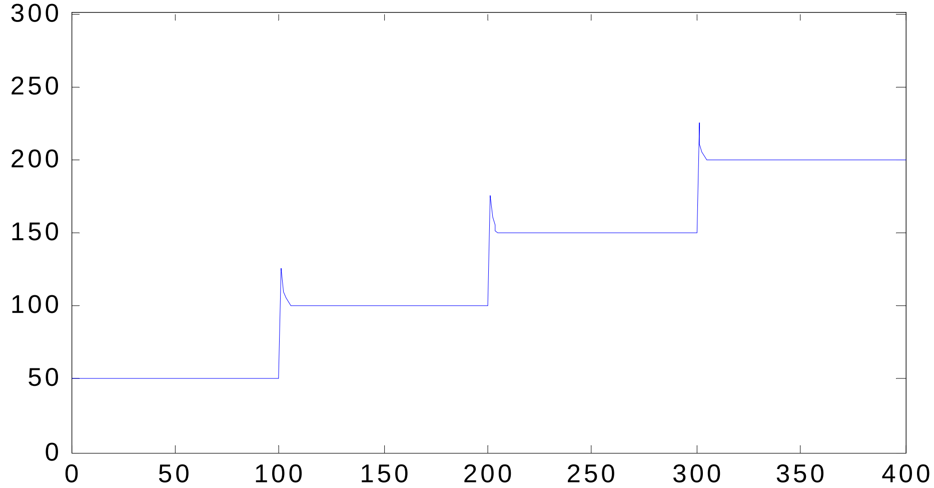
$H(z)$: $v-th$ order differentiator

$\Rightarrow$ Not same as the Mach band effect.

Fractional Calculus- Application



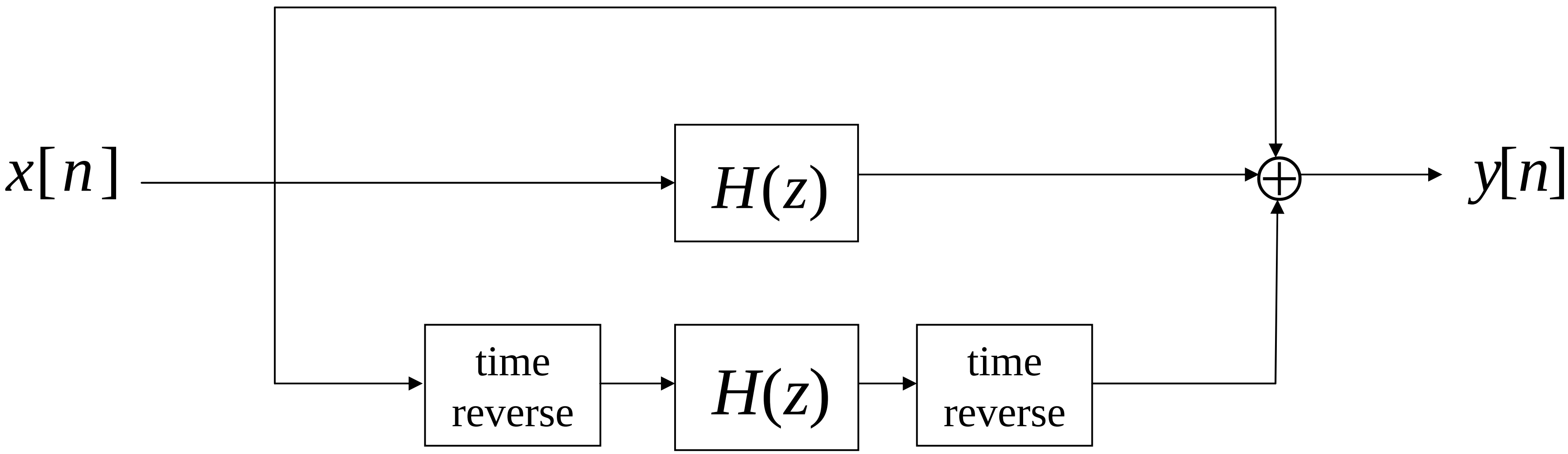
$x[n]$



$y[n]$ with
$\nu = 0.3$

Fractional Calculus - Application

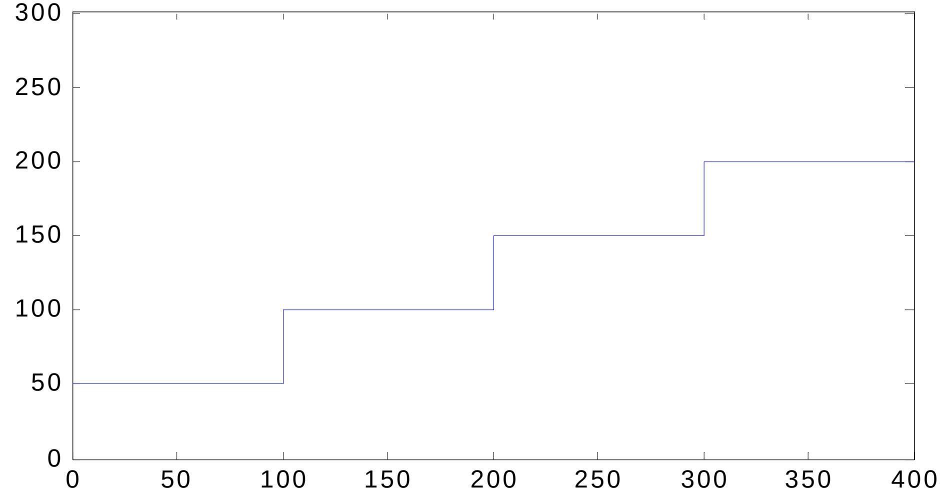
Generate Mach band effect using two fractional order differentiators



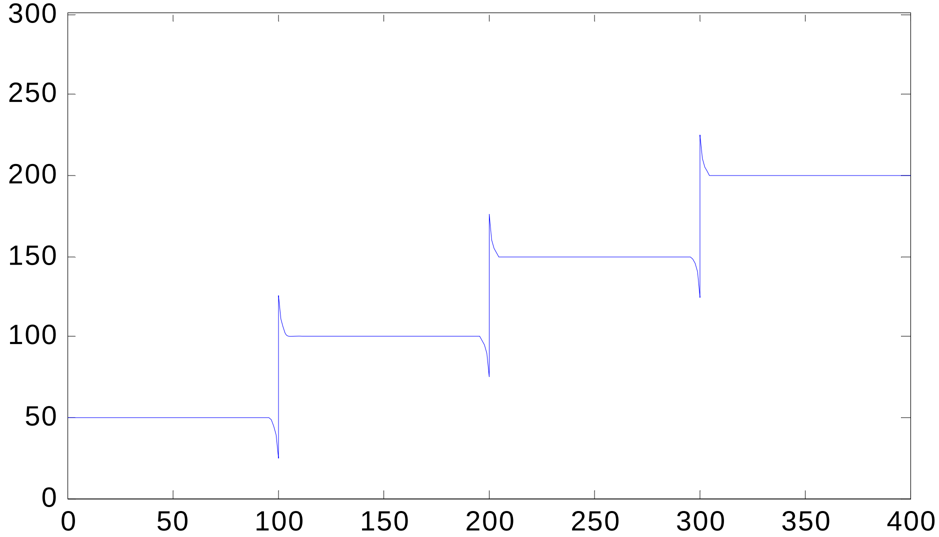
$H(z)$: $v-th$ order differentiator

$\Longrightarrow$ Same as the Mach band effect.

Fractional Calculus- Application

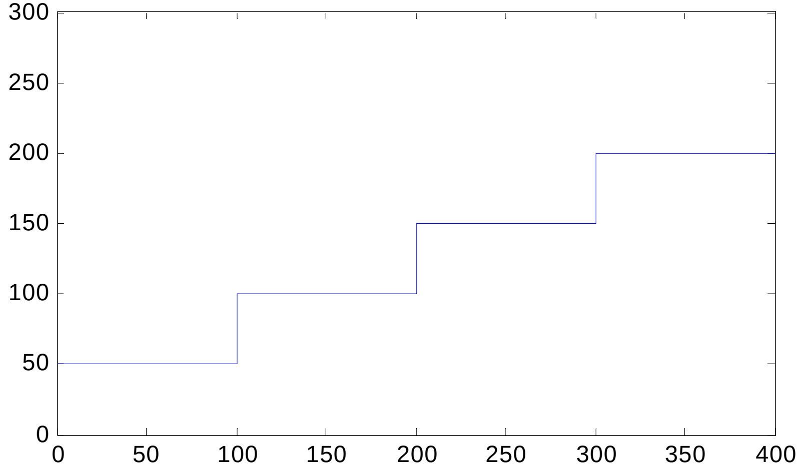


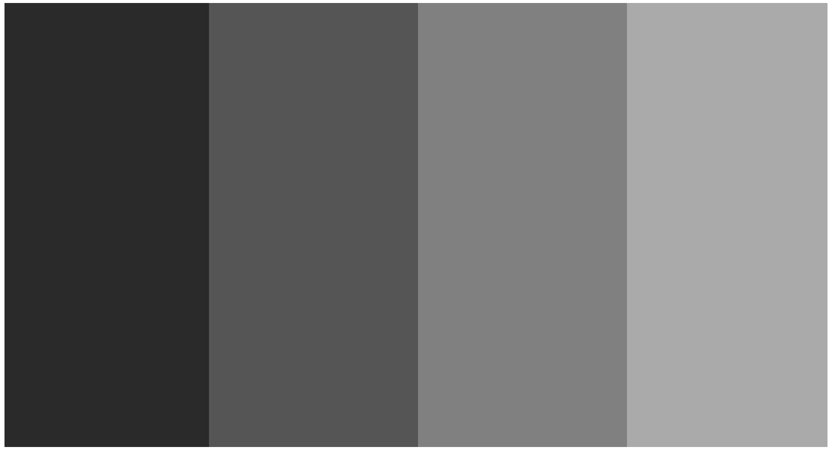
$x[n]$

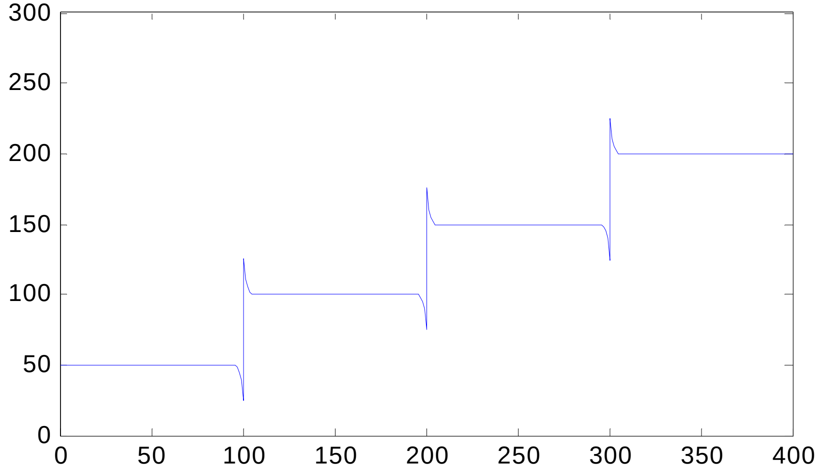


$y[n]$ with $\nu = 0.3$

Fractional Calculus- Application



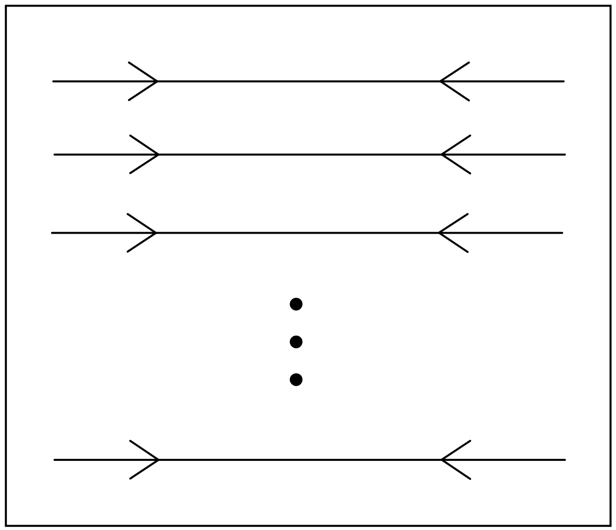




Fractional Calculus - Application

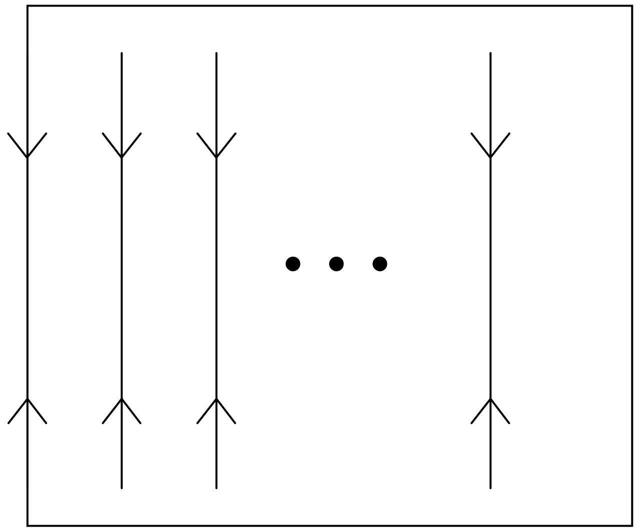
1-D $\longrightarrow$ 2-D :

image



First: row by row

image



Second: column by column

Fractional Calculus- Application

original image

$\nu = 0.3$

Fractional Calculus- Application

$\nu = 0.6$

⇨

$\nu = 1$

Fractional Calculus- Application

Conventional Laplacian Method

original image

$$f(m,n)$$

$$\downarrow$$

sharpened image

$$p(m,n) = f(m,n) + \nabla^2 f(m,n)$$

Laplacian

$$\nabla^2 f(m,n) = -f(m-1,n) - f(m,n-1) + 4f(m,n) - f(m,n+1) - f(m+1,n)$$

Fractional Calculus- Application

original image

Laplacian method

Fractional Calculus- Application

$\nu = 0.3$

Laplacian method

Fractional Calculus- Application

original image

Fractional Calculus- Application

$\nu = 0.3$

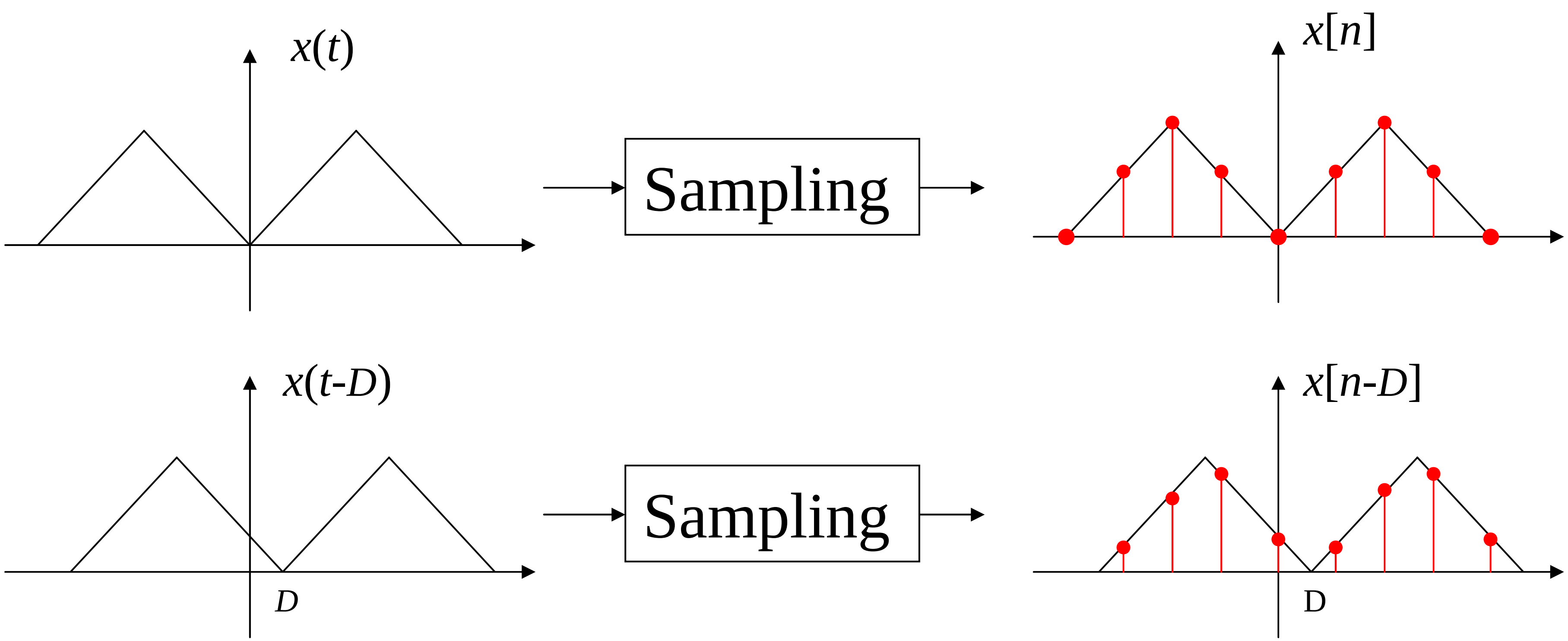
Fractional Calculus- Application

$\nu = 0.6$

Fractional Calculus- Application

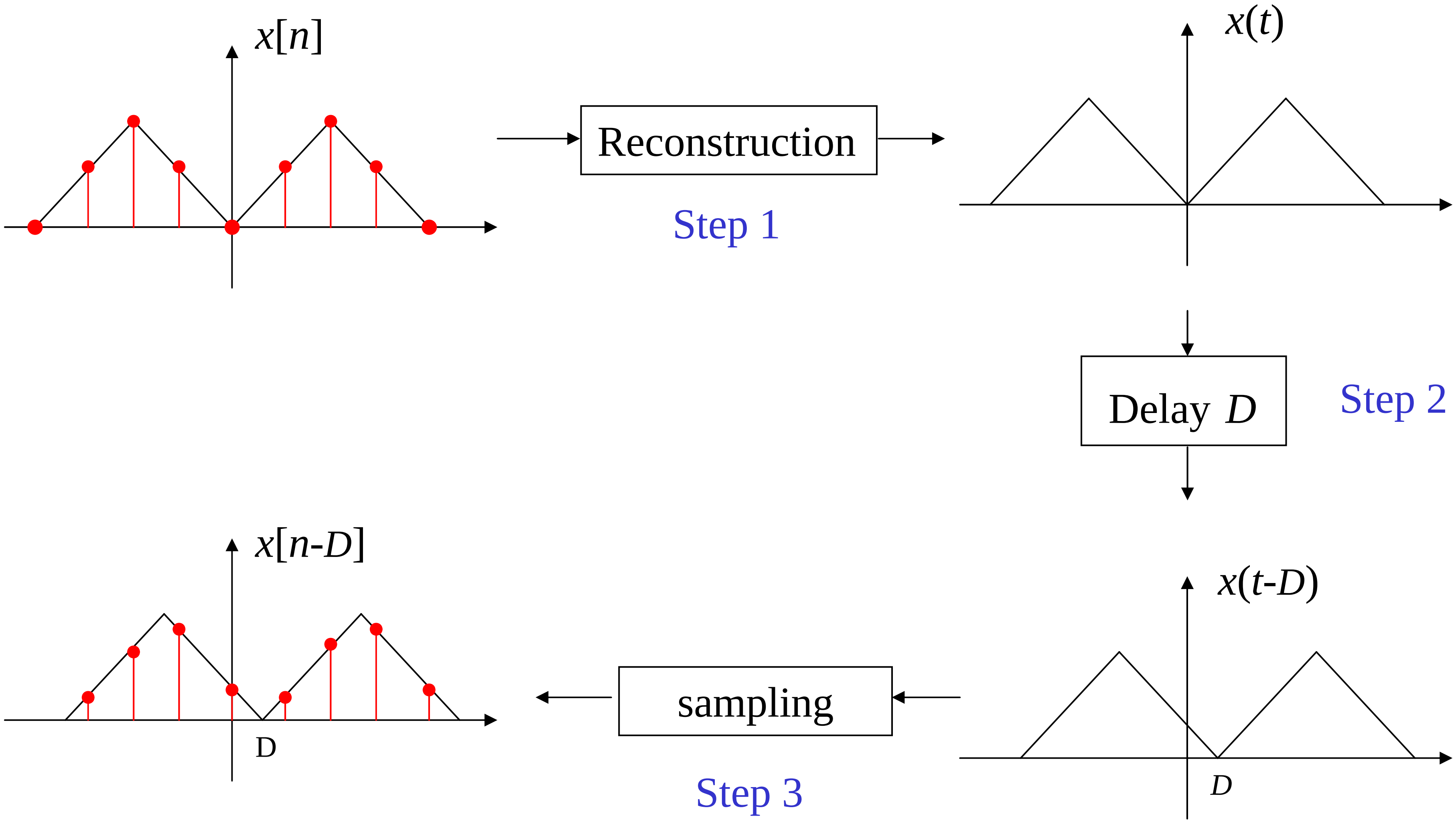
$\nu = 1$

Fractional Delay-Problem

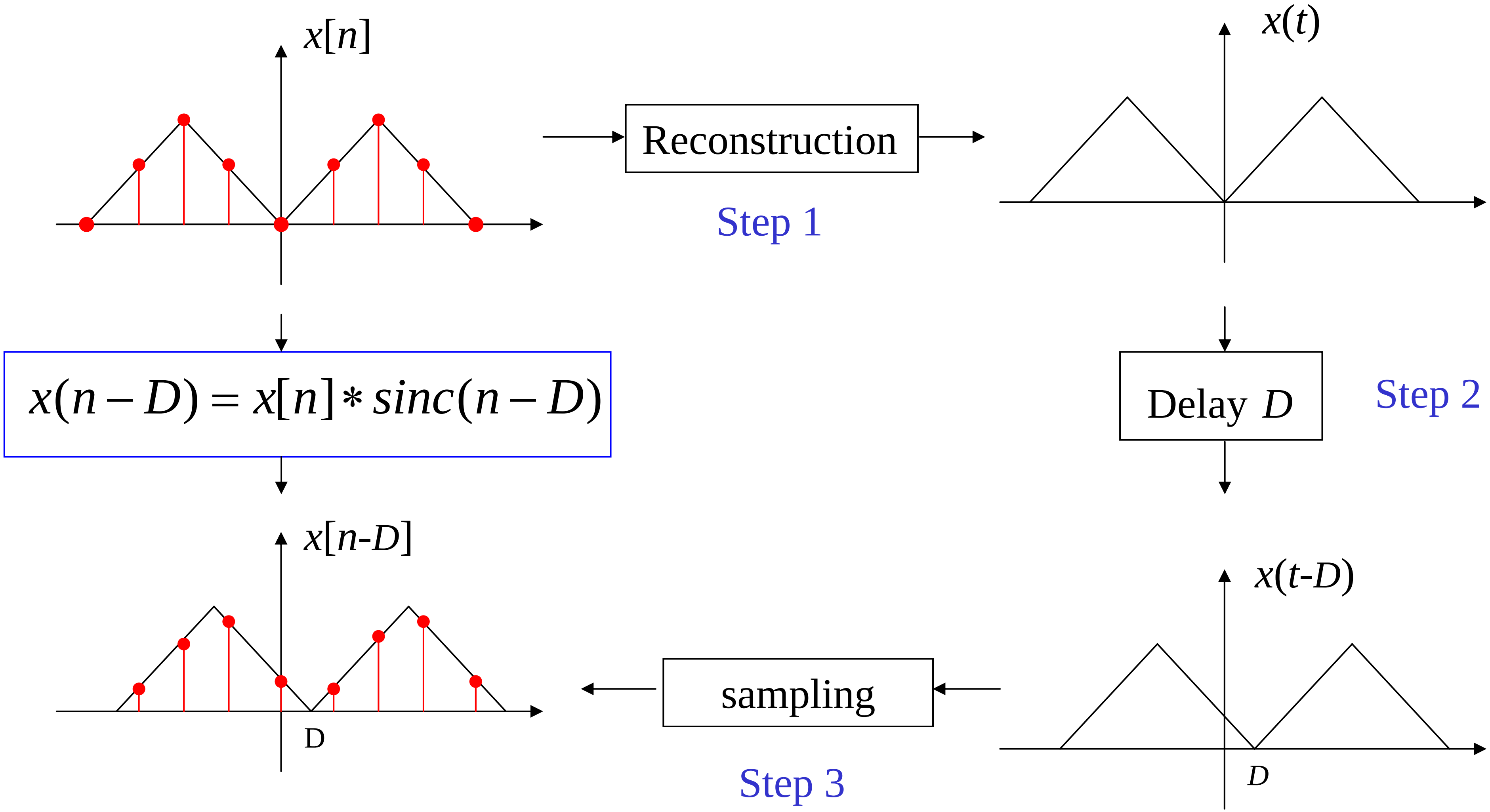


The problem is how to obtain $x[n\text{-}D]$ from $x[n]$ directly, where D is a fractional number.

Fractional Delay-Method



Fractional Delay-Method



Fractional Delay-Method

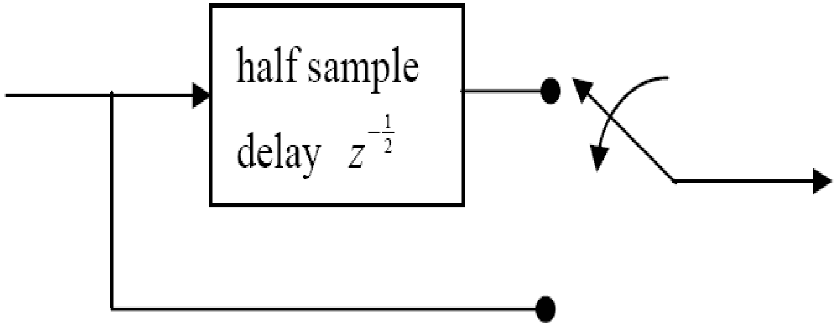
Fractional delay problem reduces to a digital filtering problem below:

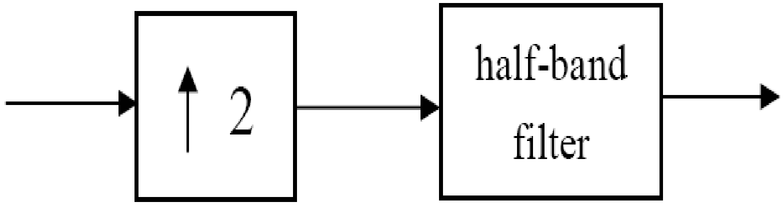
$$x[n] \rightarrow \boxed{\text{Digital filter with impulse response } sinc(n\text{-}D)} \rightarrow x[n\text{-}D]$$

$$x[n-D] = x[n] * sinc(n-D)$$

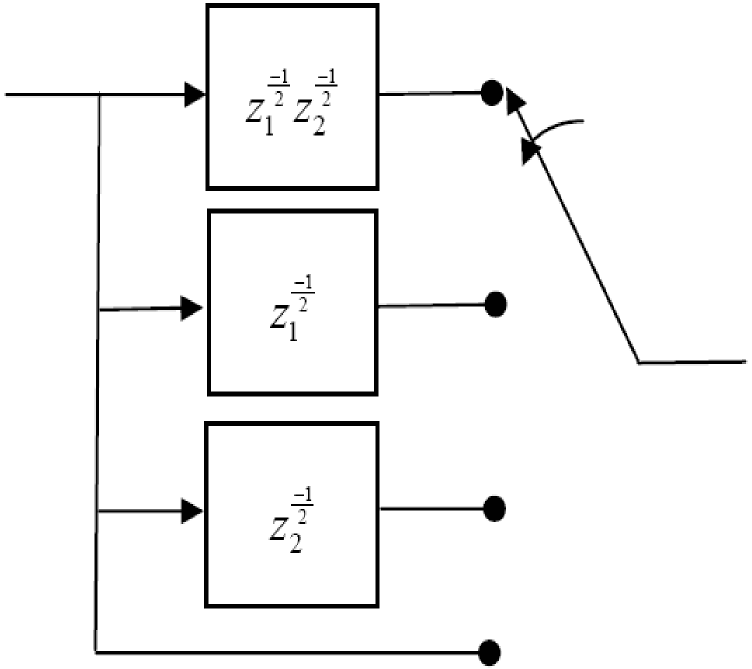
Fractional Delay-Application

1-D:





2-D:

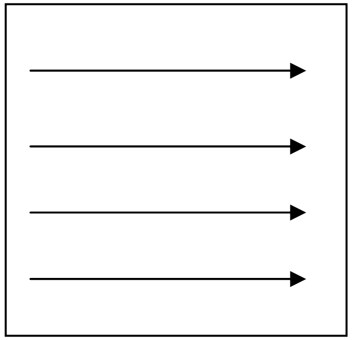


Half-sample delay for doubling sampling rate

Fractional Delay-Application

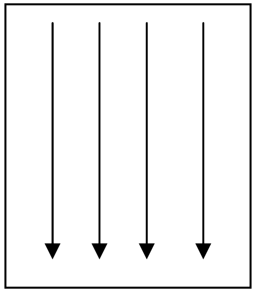
256 x 256

Fractional Delay-Application



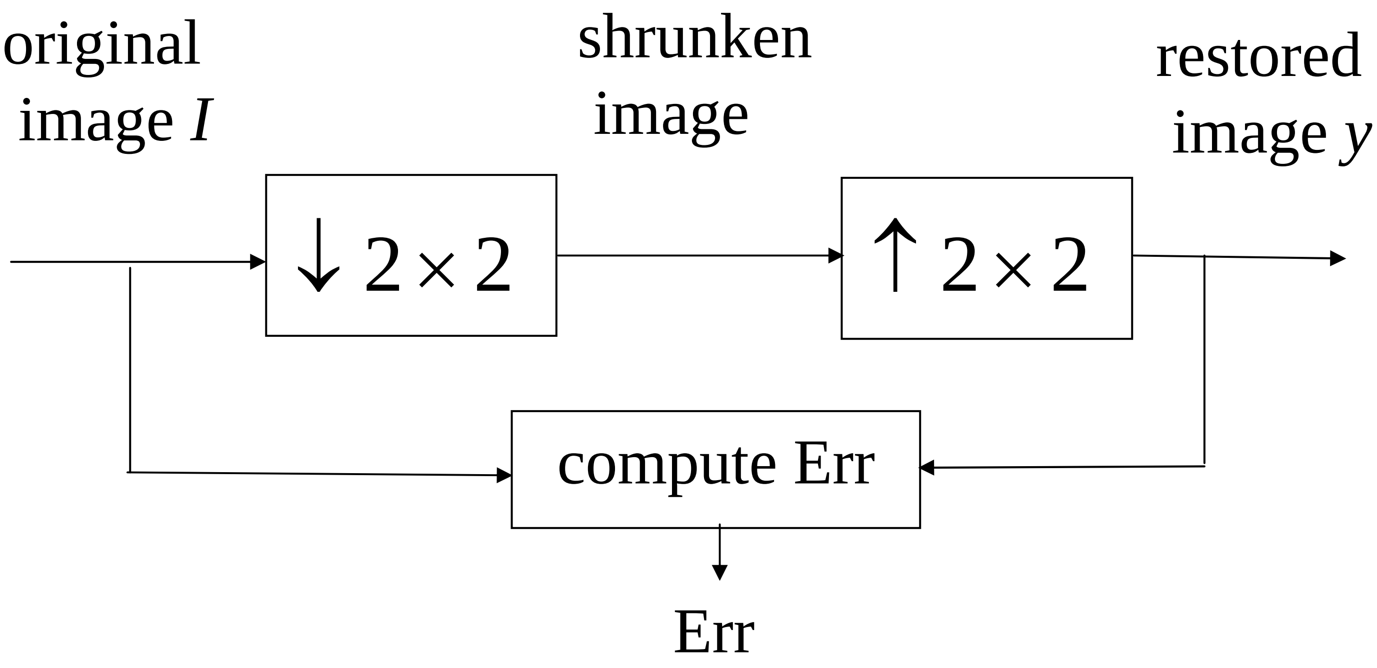
256 x 512

Fractional Delay-Application



512 x 512

Fractional Delay-Application



$$Err = \left(\frac{\sum_{n_1=1}^{512}\sum_{n_2=1}^{512}\left[I(n_1,n_2) - y(n_1,n_2)\right]^2}{\sum_{n_1=1}^{512}\sum_{n_2=1}^{512}\left[I(n_1,n_2)\right]^2} \right)^{\frac{1}{2}} \times 100\ \%$$

Fractional Delay-Application

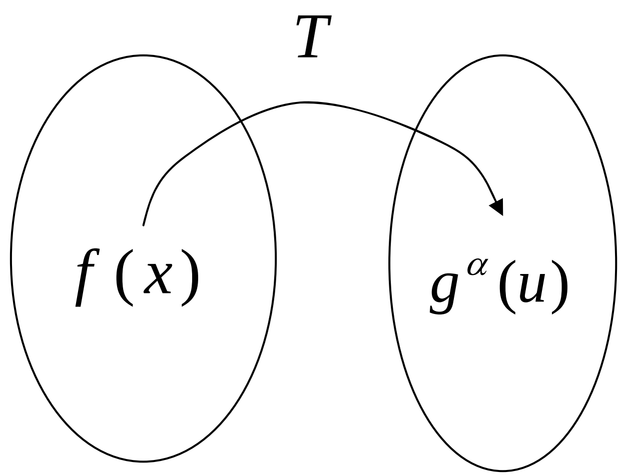
The NRMS errors *Err* in the image magnification experiment.

	Nearest neighbor	Bilinear method	Fractional delay
Lena	7.3574 %	6.3303 %	4.6149 %
Mandrill	17.8948 %	16.0415 %	14.2048 %
Lake	10.5238 %	9.5119 %	7.8363 %
Airplane	6.9903 %	5.7866 %	5.1594 %
Aerial	10.2050 %	8.5044 %	6.4829 %

Fractional Fourier Transform (1)

Fractional Fourier transform:

$$g^{\alpha}(u) = \int_{-\infty}^{\infty} f(x) K^{\alpha}(x,u)\,dx$$



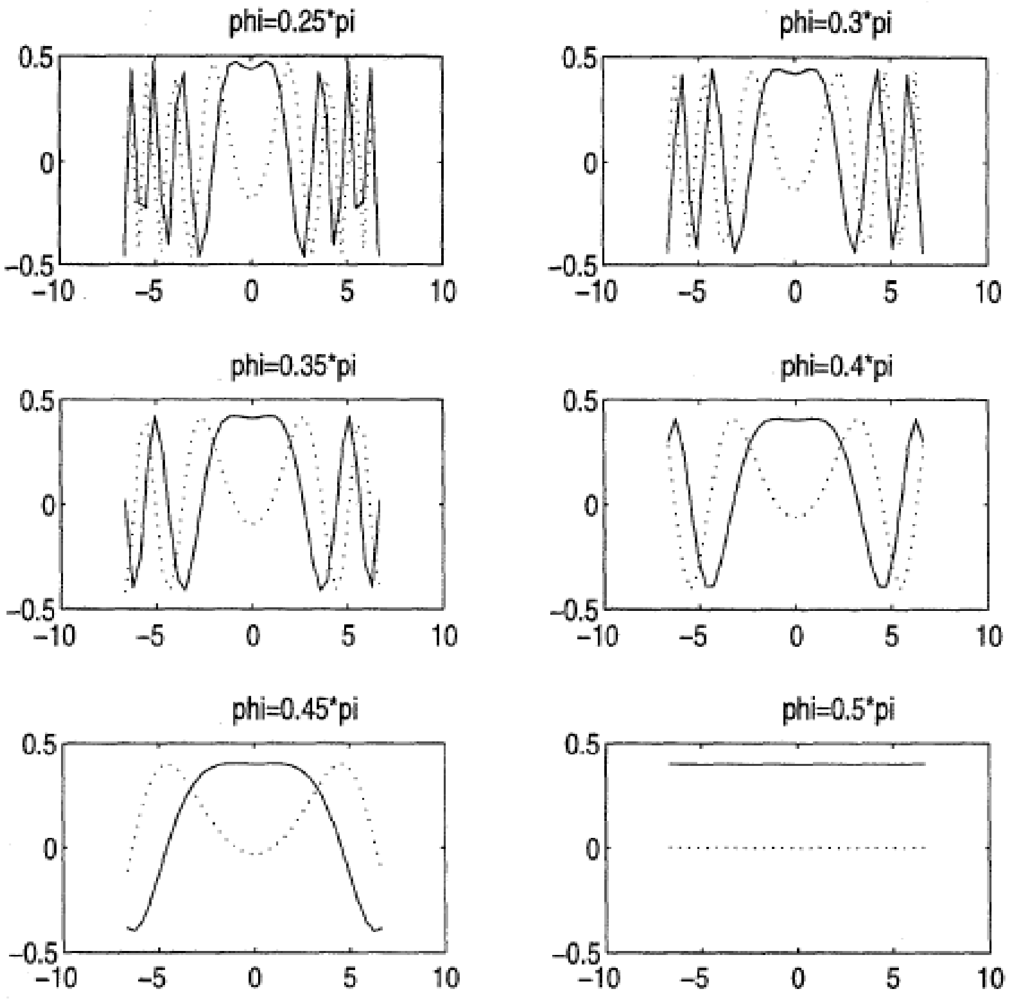
where

$$K^{\alpha}(x,u) = \sqrt{\frac{1-j\cot\phi}{2\pi}}\, e^{j\frac{x^2+u^2}{2}\cot\phi - jxu\csc\phi}$$

Chirp basis

$$\phi = \frac{\alpha\pi}{2}$$

Fractional Fourier Transform (2)



$$f(x) = \delta(x)$$

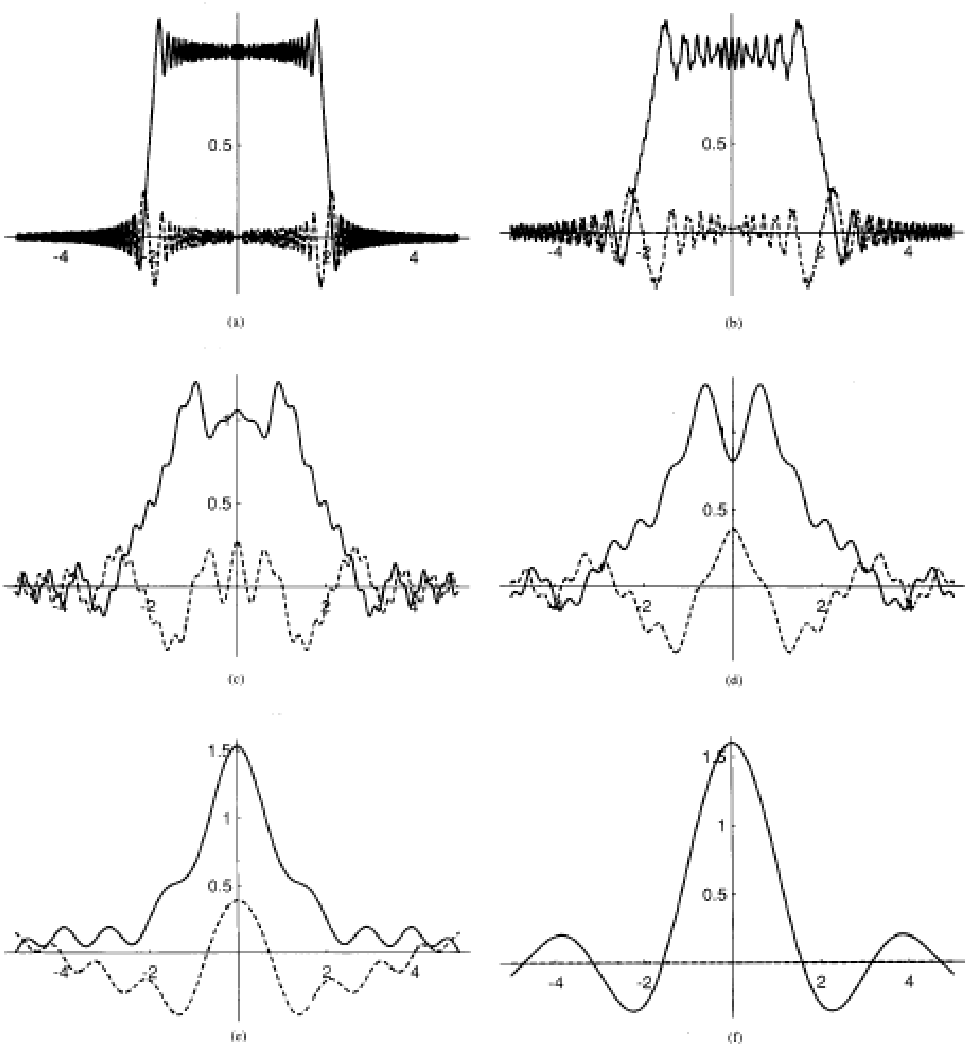
$\downarrow$ FrFT

$$T^{\alpha}(f(x))$$

$$= g^{\alpha}(u)$$

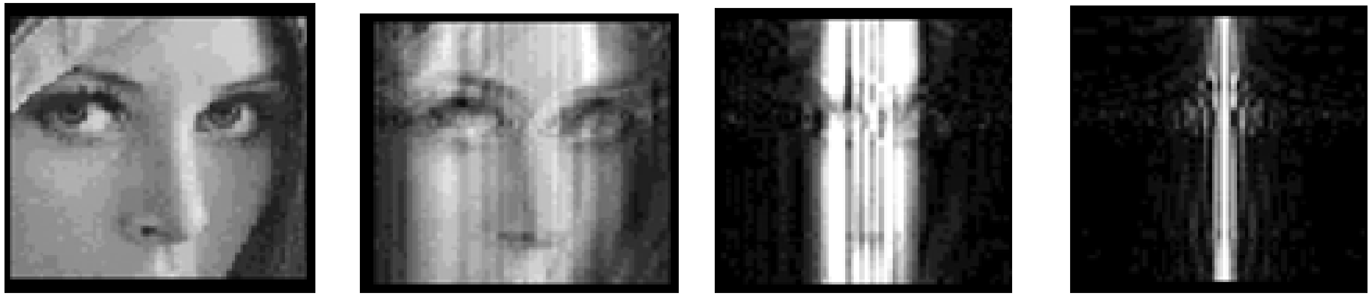
$$= \sqrt{\frac{1 - j\cot\phi}{2\pi}}\, e^{j\frac{u^2}{2}\cot\phi}$$

Fractional Fourier Transform (3)

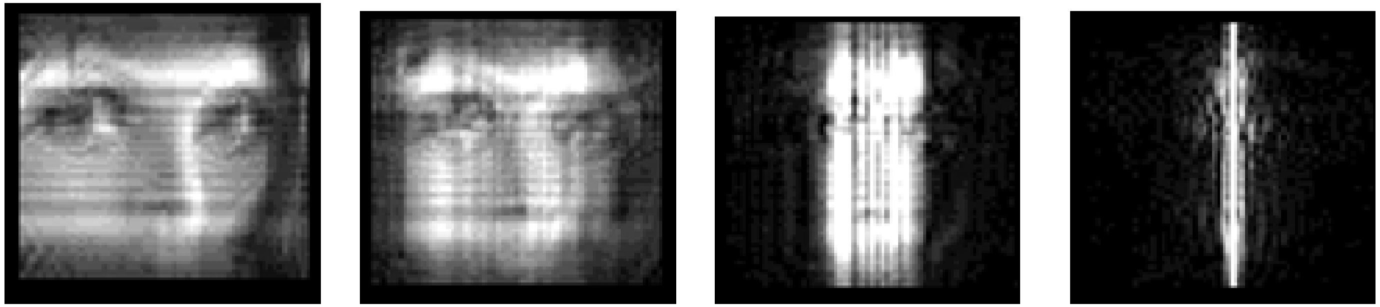


Fractional Fourier Transform (4)

$\alpha = 0,\ \ \beta = 0$ $\alpha = \frac{\pi}{6},\ \ \beta = 0$ $\alpha = \frac{\pi}{3},\ \ \beta = 0$ $\alpha = \frac{\pi}{2}, \beta = 0$

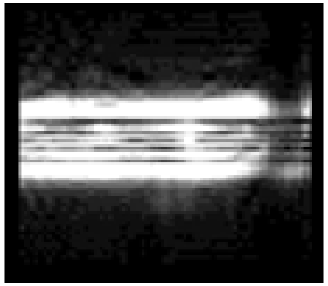


$\alpha = 0,\ \ \beta = \frac{\pi}{6}$ $\alpha = \frac{\pi}{6},\ \ \beta = \frac{\pi}{6}$ $\alpha = \frac{\pi}{3},\ \ \beta = \frac{\pi}{6}$ $\alpha = \frac{\pi}{2},\ \ \beta = \frac{\pi}{6}$

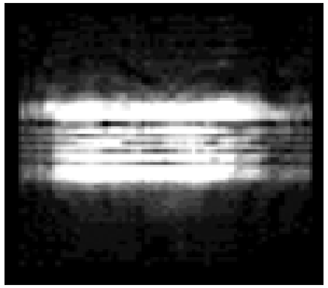


Fractional Fourier Transform (5)

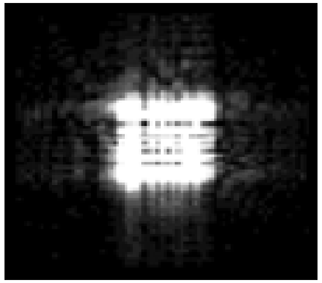
$\alpha = 0,\ \ \beta = \frac{\pi}{3}$



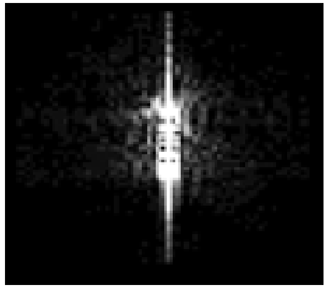
$\alpha = \frac{\pi}{6},\ \ \beta = \frac{\pi}{3}$



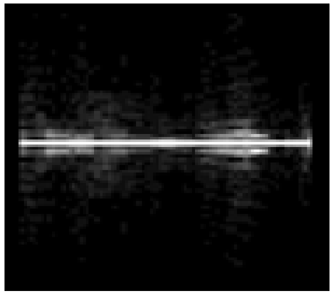
$\alpha = \frac{\pi}{3},\ \ \beta = \frac{\pi}{3}$



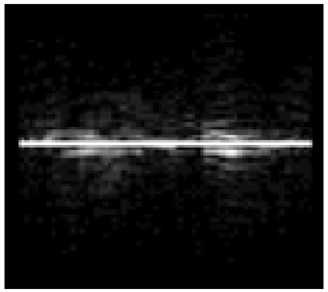
$\alpha = \frac{\pi}{2},\ \ \beta = \frac{\pi}{3}$



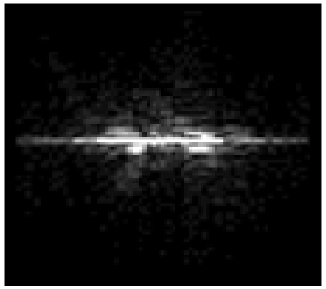
$\alpha = 0,\ \ \beta = \frac{\pi}{2}$



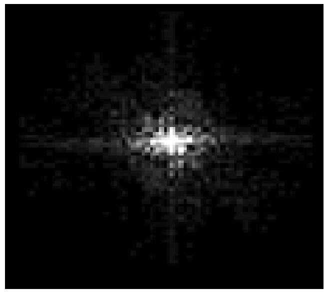
$\alpha = \frac{\pi}{6},\ \ \beta = \frac{\pi}{2}$



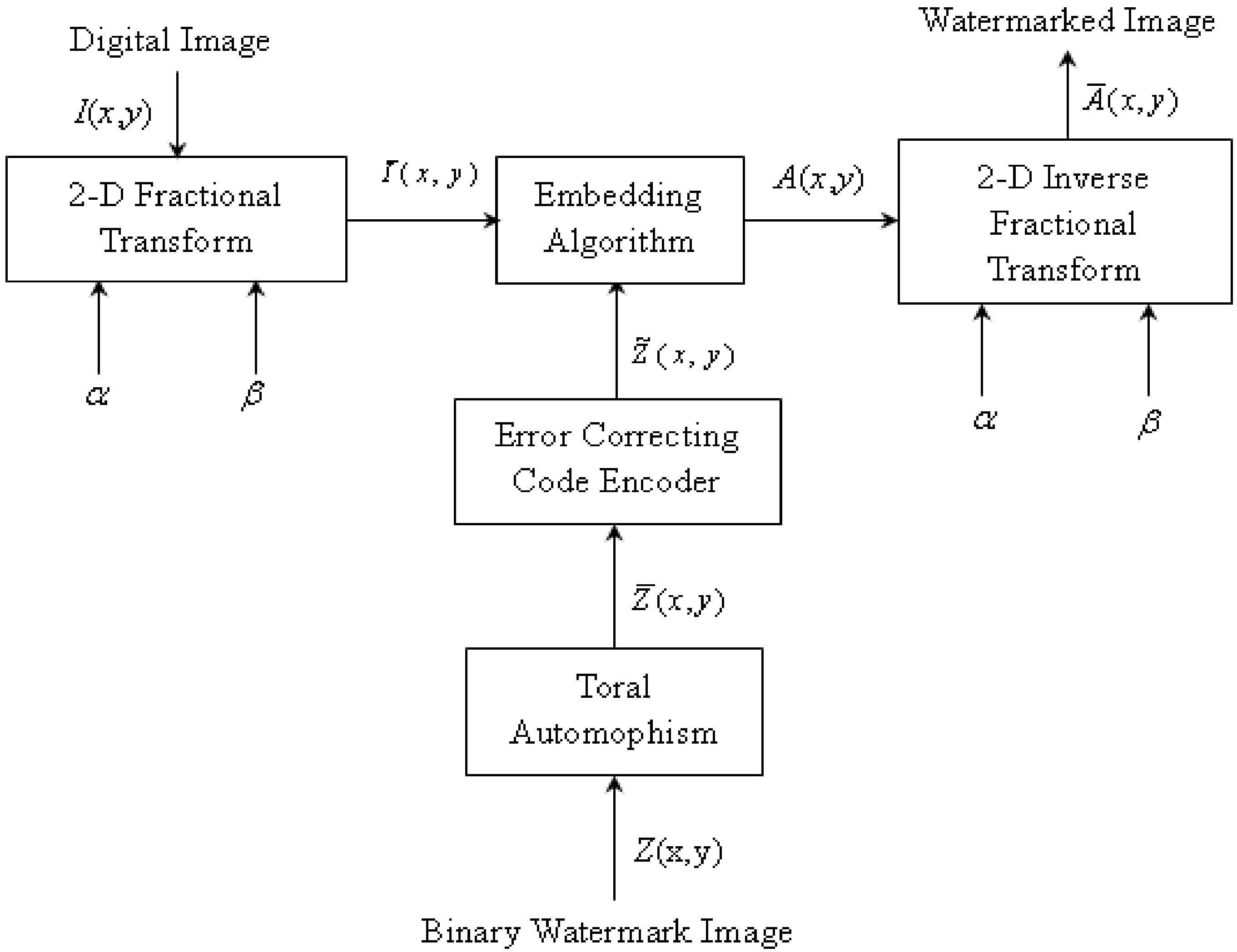
$\alpha = \frac{\pi}{3},\ \ \beta = \frac{\pi}{2}$



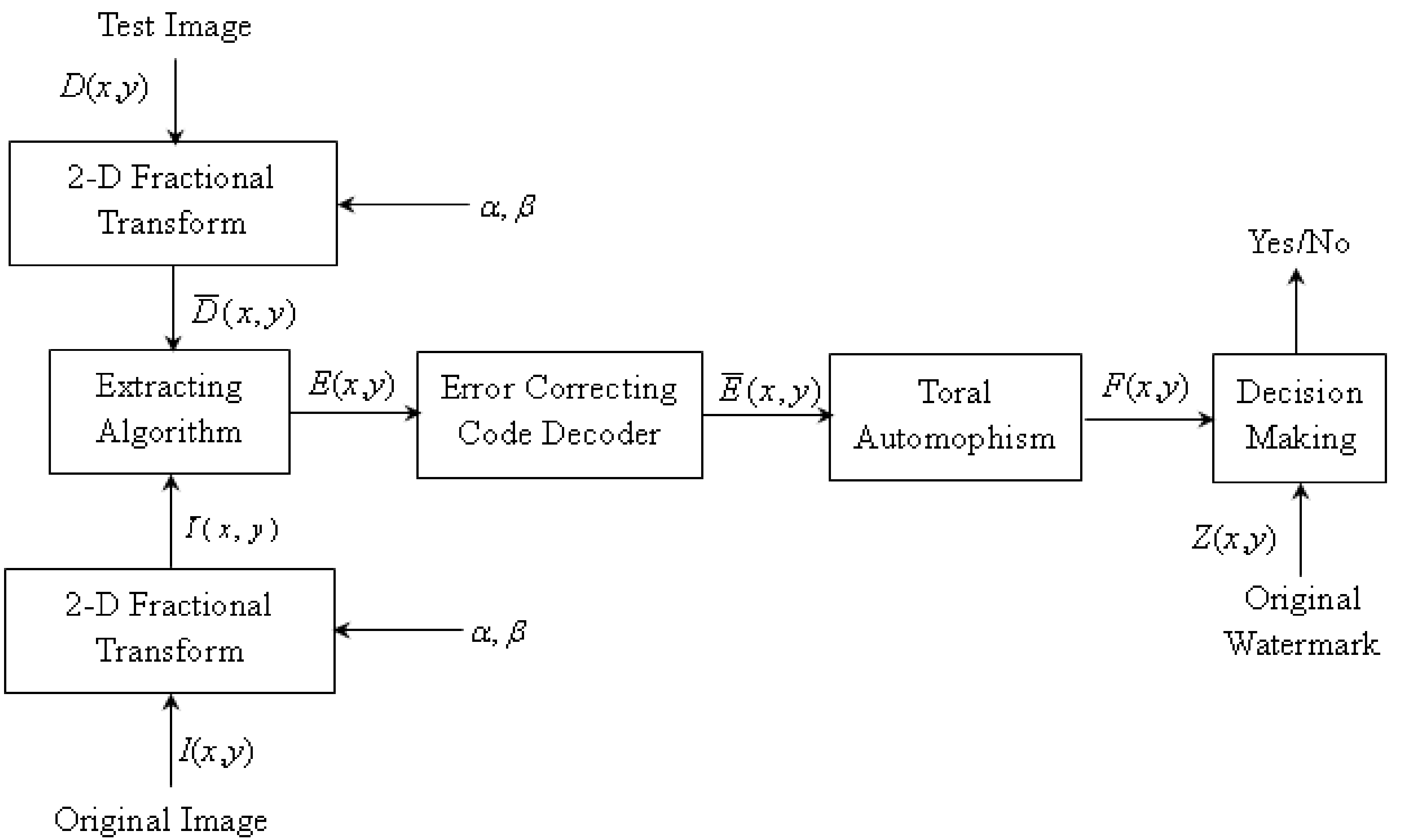
$\alpha = \frac{\pi}{2},\ \ \beta = \frac{\pi}{2}$



Fractional Fourier Transform (6)



Fractional Fourier Transform (7)



Fractional Fourier Transform (8)

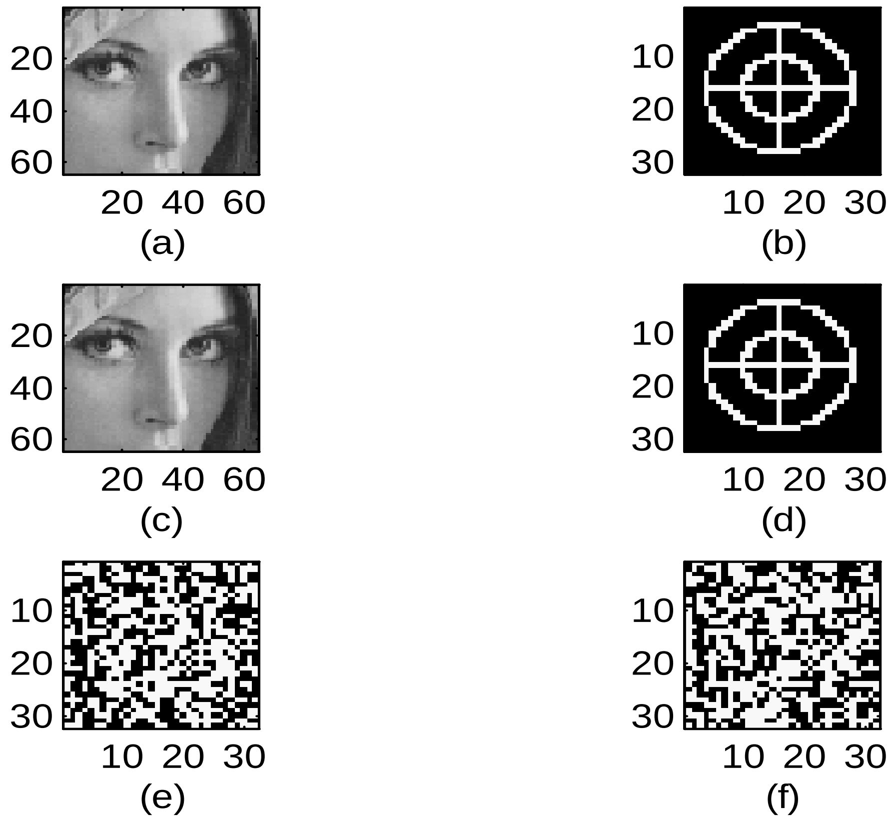
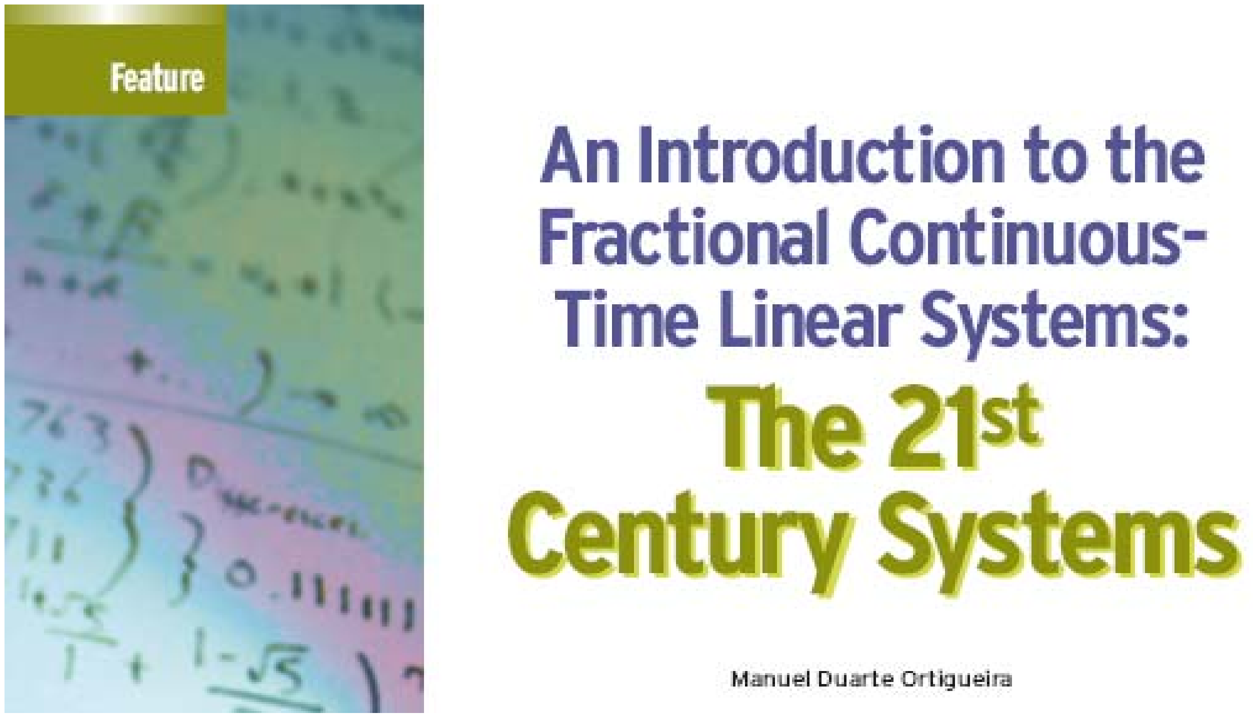


Image watermarking in the fractional DFT domain. (a) Original image (b) Original watermark (c) Watermarked image with (α,β)=(0.3,0.3) (d) Extracted watermark with (α,β)=(0.3,0.3) (e) Extracted watermark with (α,β)=(0,0) (f) Extracted watermark with (α,β)=(0.5,0.7).

Conclusion

Invite you to join fractional world !!!



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